DU Mathsoc Problem-Solving Week 9

December 2nd, Michaelmas 2021

Problem 1: *What is the value of* $\dfrac{11! - 10!}{9!}$ *?*

(AMC 10A 2016)

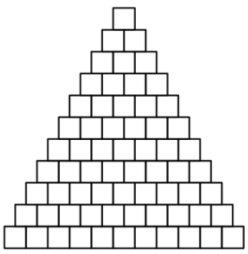
Problem 2: *An* 8 *by* $2\sqrt{2}$ *rectangle has the same center as a circle of radius* 2*. What is the area of the region common to both the rectangle and the circle?*

(AHSME 1994)

Problem 3: *How many two-digit primes have both their digits non-prime?*

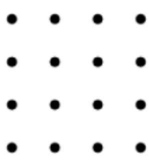
(JMC 2020)

Problem 4: *A triangular array of squares has one square in the first row, two in the second, and in general,* k *squares in the* k*th row for* $1 \leq k \leq 11$*. With the exception of the bottom row, each square rests on two squares in the row immediately below (illustrated in given diagram). In each square of the eleventh row, a* 0 *or a* 1 *is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of* 0*'s and* 1*'s in the bottom row is the number in the top square a multiple of* 3*?*



(AIME 2007)

Problem 5: *The diagram below shows a* 4×4 *rectangular array of points, each of which is* 1 *unit away from its nearest neighbors.*



Define a growing path to be a sequence of distinct points of the array with the property that the distance between consecutive points of the sequence is strictly increasing. Let m *be the maximum possible number of points in a growing path, and let* r *be the number of growing paths consisting of exactly* m *points. Find* m *and* r*.*

(AIME 2008)

Problem 6: *Determine all pairs* (a, b) *of real numbers for which there exists a unique symmetric* 2×2 *matrix* M *with real entries satisfying trace(*M*)=* a *and det(*M*)=* b*.*

(IMC 2014)

Problem 7: *Alan and Barbara play a game in which they take turns filling entries of an initially empty* 2008×2008 *array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?*

(Putnam 2008 A2)

Problem 8: *Let* $f(x)$ *be a real valued function defined for all* $x \geq 1$*, satisfying* $f(1) = 1$ *and*

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

Prove that

$$\lim_{x \to \infty} f(x)$$

exists and is less than $1 + \frac{\pi}{4}$*.*

(USAMO 2002)

Problem 9: *A sphere is colored in two colors. Prove that there exist on this sphere three points of the same color, which are vertices of a regular triangle.*

(Problem-Solving Strategies)