DU Mathsoc Problem-Solving Week 8

November 25th, Michaelmas 2021

Problem 1: *The five-digit integer $a679b$ is a multiple of 72. What are the values of a and b?*

(Hamilton Olympiad 2015)

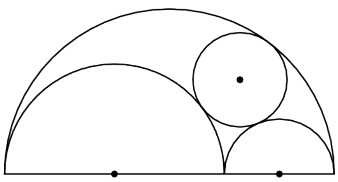
Problem 2: *Consider the triangular array of numbers with 0,1,2,3,... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.*

0
1 1
2 2 2
3 4 4 3
4 7 8 7 4
5 11 15 15 11 5

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

(AHSME 1995)

Problem 3: *In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter JK. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?*



(AMC 12A 2017)

Problem 4: *Let B be the set of all binary integers that can be written using exactly 5 zeros and 8 ones where leading zeros are allowed. If all possible subtractions are performed in which one element of B is subtracted from another, find the number of times the answer 1 is obtained.*

Problem 5: *One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?*

(BMO1 2011)

Problem 6: *Find, with proof, all pairs of integers (x, y) satisfying the equation*

$$1 + 1996x + 1998y = xy$$

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(IRMO 1997)

Problem 7: *Find the least positive integer N such that the set of 1000 consecutive integers beginning with $1000 \cdot N$ contains no square of an integer.*

(AIME 2013)

Problem 8: *Is it true that if $f : [0, 1] \to [0, 1]$ is*
(a) monotone increasing
(b) monotone decreasing
then there exists an $x \in [0, 1]$ for which $f(x) = x$?

(IMC 2000)

Problem 9: *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - y^2) = xf(x) - yf(y)$ over $\mathbb{R}$.*

(USAMO 2002)

Problem 10: *Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?*

(IMC 2007)

Problem 11: *In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?*

(Putnam A4 2002)