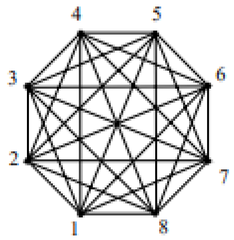
DU Mathsoc Problem-Solving Week 7

November 18th, Michaelmas 2021

Problem 1: *In the figure shown, each line joining two numbers is to be labelled with the sum of the two numbers that are at its end points. How mnay of these labels are multiples of 3?*

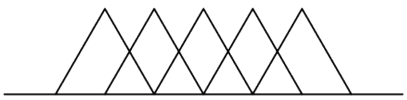


(JMC 2015)

Problem 2: *A positive integer is called ascending if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?*

(AIME 1992)

Problem 3: *Five equilateral triangles, each with side $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each vertex. Along this line, the midpoint of the base of one triangle is a vertex of the next. Find the area of the region of the plane that is covered by the union of the five triangular regions.*



(AHSME 1992)

Problem 4: *The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and*

$$f(x) + f\left(\frac{1}{x}\right) = x$$

What is the largest set of real numbers that can be in the domain of f?

(AMC 12A 2006)

Problem 5: *Find all real numbers x, y, z such that $x^2 + y^2 + z^2 = x - z = 2$.*

(Hamilton Olympiad 2021)

Problem 6: *Find, with proof, all triples of integers (a, b, c) such that a, b and c are the lengths of the sides of a right angled triangle whose area is $a + b + c$.*

(IRMO 2008)

Problem 7: *For each positive integer n, let*

$$a_n = \frac{(n+9)!}{(n-1)!}$$

Let k denote the smallest positive integer for which the rightmost nonzero digit of a_k is odd. What is the rightmost nonzero digit of a_k?

(AHSME 1998)

Problem 8: *Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?*

(Putnam 2009 A1)

Problem 9: *Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that*

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

(IMO 1988 P6)

Problem 10: *Suppose that $n > 1$ is an integer. Prove that the sum*

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

(Berkeley Problems in Mathematics)