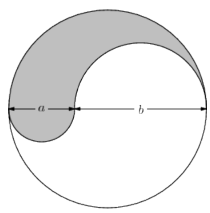
DU Mathsoc Problem-Solving Week 6

November 11th, Michaelmas 2021

Problem 1: *The figure shown is the union of a circle and two semicircles of diameters a and b, all of whose centers are collinear. Find the ratio of the area of the shaded region to that of the unshaded region.*



(AHSME 1998)

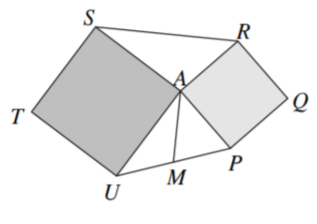
Problem 2: *A pair of standard 6-sided dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?*

(AMC 12A 2011)

Problem 3: *Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.*

(AIME 2015)

Problem 4: *The diagram shows two squares $APQR$ and $ASTU$, which have vertex A in common. The point M is the midpoint of PU.*
Prove that $AM = \frac{1}{2}RS$.



(Andrew Smith)

Problem 5: *For each positive integer n, let $f(n)$ denote the highest common factor of $n! + 1$ and $(n + 1)!$. Find, with proof, a formula for $f(n)$ for each n.*

Problem 6: *Mary and Pat play the following number game. Mary picks an initial integer greater than 2017. She then multiplies this number by 2017 and adds 2 to the result. Pat will add 2019 to this new number and it will again be Mary's turn. Both players will continue to take alternating turns. Mary will always multiply the current number by 2017 and add 2 to the result when it is her turn. Pat will always add 2019 to the current number when it is his turn. Pat wins if any of the numbers obtained by either player is divisible by 2018. Mary wants to prevent Pat from winning the game. Determine, with proof, the smallest initial integer Mary could choose in order to achieve this.*

(IRMO 2018)

Problem 7: *Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.*
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

(AIME 2012)

Problem 8: *Let $0 < a < b$. Prove that:*

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq e^{-a^2} - e^{-b^2}.$$

(IMC 2010)

Problem 9: *Find a square root of the matrix*

$$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}$$

How many square roots does this matrix have?

(Berkeley Problems in Mathematics (7.6.12))

Problem 10: *Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.*

(Putnam 2017 B3)