DU Mathsoc Problem-Solving Week 5

November 4th, Michaelmas 2021

Problem 1: *In a magic square, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.*

x	19	96
1		

(AIME 1996)

Problem 2: *Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?*

(AMC 12A 2017)

Problem 3: *A triangular playground has sides, in metres, measuring 7, 24 and 25. Inside the playground, a lawn is designed so that the distance from each point on the edge of the lawn to the nearest side is 2 metres. What is the area of the lawn?*

(Maclaurin Olympiad 2021)

Problem 4: *How many zeros does $f(x) = \cos(\log x)$ have on the interval $0 < x < 1$?*

(AHSME 1999)

Problem 5: *Place the following numbers in increasing order of size, and jusitfy your reasoning:*

$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3}, 4^{3^4}$$

(BMO1 2015)

Problem 6: *Prove that*

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

for every positive integer n.

(Problem-Solving Methods in Combinatorics)

Problem 7: *A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following for all $n \in \mathbb{N}$:*

$$f(1) = 1$$
$$f(f(n)) = n$$
$$f(2n) = 2f(n) + 1$$

Find the value of $f(2020)$.

(IRMO 2020)

Problem 8: *A curve C partitions the area of a parallelogram into two equal parts.*
Prove that there exist two points A, B of C such that the line AB passes through the center O of the parallelogram.

(Problem-Solving Strategies)

Problem 9: *Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:*

(1) $det(B) = 1$;

(2) $AB = BA$;

(3) $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(IMC 2019)

Problem 10: *Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is*

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

(Putnam 2001 B1)