DU Mathsoc Problem-Solving Week 4

October 21st, Michaelmas 2021

Problem 1: *What is the largest number of acute angles that a convex hexagon can have?*

(AHSME 1999)

Problem 2: *In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0$. What is the greatest distance between a point of A and a point of B?*

(AMC 12A 2020)

Problem 3: *Alice randomly chooses a number A from $1, 2, 3, \ldots, 19, 20$. Bob then randomly chooses a number B from $1, 2, 3, \ldots, 19, 20$ distinct from A. What is the probability that the value of $B - A$ is at least 2?*

(AIME 2019)

Problem 4: *Prove that $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.*

(IMO 1959)

Problem 5: *Prove that if p is prime and $0 < k < p$, then $\binom{p}{k}$ is divisible by p.*

Bonus: prove that $\binom{n}{k}$ is divisble by n for all $0 < k < n$ if and only if n is prime.

(Problem-Solving Methods in Combinatorics)

Problem 6: *The floor of $x \in \mathbb{R}$ is the unique integer $\lfloor x \rfloor$ such that $x = \lfloor x \rfloor$, with $0 \leq r < 1$. Find all values of x for which $\lfloor 2x \rfloor = x^2 + x$.*

(John Murray)

Problem 7: *Let $n \geq 2$ be a positive integer and let $S = \{1, 2, ..., n\}$. Find the number of non-decreasing functions $f : S \to S$ such that*

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in S$.

(Mark Flanagan)

Problem 8: *Consider a circle of radius 1, and let $Q_1, Q_2, \ldots, Q_n$ be the vertices of a regular n-gon inscribed in the circle. Join Q_1 to $Q_2, Q3, \ldots, Q_n$ by segments of a straight line. You obtain $(n-1)$ segments of lengths $\lambda_2, \lambda_3, \ldots, \lambda_n$. Show that*

$$\prod_{i=2}^{n} \lambda_i = n$$

(Berkeley Problems in Mathematics)

Problem 9: *Let A be a real $n \times n$ matrix such that $A^3 = 0$.*
(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(IMC 2021)

Problem 10: *For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as*

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

(IMO 2017)

Problem 11: *Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color this neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find*

$$\lim_{N \to \infty} \frac{w(N)}{N}.$$

(Putnam 2020 A4)