DU Mathsoc Problem-Solving Week 3

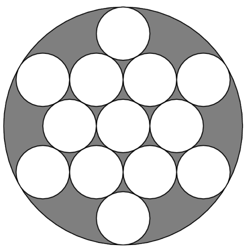
October 14th, Michaelmas 2021

Problem 1: *How many different prime numbers are factors of N if*

$$\log_2(\log_3(\log_5(\log_7 N))) = 11?$$

(AHSME 1998)

Problem 2: *The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?*



(AMC 10A 2019)

Problem 3: *The expressions $A = 1\times 2+3\times 4+5\times 6+\cdots+37\times 38+39$ and $B = 1+2\times 3+4\times 5+\cdots+36\times 37+38\times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B.*

(AIME 2015)

Problem 4: *Let $n = 2^{33}3^{17}$. How many positive integer divisors of n^2 are less than n but do not divide n?*

(Mark Flanagan)

Problem 5: *Let p be a prime number. Show that*

$$\binom{p-1}{r} \equiv (-1)^r \textit{ mod } p$$

for integers $0 \le r \le p-1$, where $\binom{p-1}{r}$ is the binomial coefficient.

(Andrew Smith)

Problem 6: *Find the number of ways to place 3 rooks on a 5×5 chess board so that no two of them attack each other.*
(Assume a rook attacks any other piece in its row and column).

(Problem-Solving Methods in Combinatorics)

Problem 7: *Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that*

$$f(x+f(y)) = y + f(x), \textit{ for all } x, y \in \mathbb{Q}$$

(IRMO 2002)

Problem 8: *A positive integer n is called 'good' if there is a set of divisors of n whose members sum to n and include 1. Prove that every positive integer has a multiple which is good.*

(BMO2 2021)

Problem 9: *Which of the following series converges?*

$$1)\quad \Sigma_{n=1}^{\infty}\frac{(2n)!(3n)!}{n!(4n)!}$$

$$2)\quad \Sigma_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$$

(Berkeley Problems in Mathematics)

Problem 10: *Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.*

(IMO 2021)

Problem 11: *Determine all possible values of the expression*

$$A^3 + B^3 + C^3 - 3ABC$$

where $A, B,$ and C are nonnegative integers.

(Putnam 2019 A1)