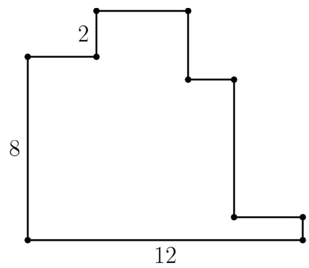
DU Mathsoc Problem-Solving Week 2

October 7th, Michaelmas 2021

Problem 1: *The adjacent sides of the decagon shown meet at right angles. What is its perimeter?*



(AHSME 1997)

Problem 2: *At a gathering of* 30 *people, there are* 20 *people who all know each other and* 10 *people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?*

(AMC 12A 2017)

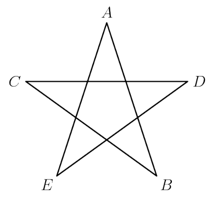
Problem 3: *There is a prime number* p *such that* $16p + 1$ *is the cube of a positive integer. Find* p.

(AIME 2015)

Problem 4: *What is the area of the region enclosed by the graph of the equation* $x^2 + y^2 = |x| + |y|$?

(AMC 12B 2016)

Problem 5: *In the five-sided star shown, the letters A, B, C, D and E are replaced by the numbers 3, 5, 6, 7 and 9, although not necessarily in that order. The sums of the numbers at the ends of the line segments* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, *and* $\overline{EA}$ *form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?*



(AMC 12A 2005)

Problem 6: *(a) Find all positive integers* n *for which* $2^n - 1$ *is divisible by* 7.
(b) Prove that there is no positive integer n *for which* $2^n + 1$ *is divisible by* 7.

(IMO 1964)

Problem 7: *If the three-digit number* ABC *is divisible by* 27, *prove that the three-digit numbers* BCA *and* CAB *are also divisible by* 27.

(IRMO 2016)

Problem 8: *A quadrilateral has one vertex on each side of a square with sides of length 1. Show that the lengths a, b, c, d of the sides of the quadrilateral satisfy the inequalities*

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$$

Problem 9: *The Fibonacci numbers* $f_1, f_2, \ldots$ *are defined re-cursively by* $f_1 = 1, f_2 = 2$, *and* $f_{n+1} = f_n + f_{n-}$ *for* $n \geq 2$. *Show that*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

exists, and evaluate the limit.

(Berkeley Problems in Mathematics)

Problem 10: *Let* $\mathbb{Z}$ *be the set of integers. Determine all functions* $f : \mathbb{Z} \to \mathbb{Z}$ *such that, for all integers* a *and* b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

(IMO 2019)

Problem 11: *Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.*

(Putnam 2013 A1)