DU Mathsoc Problem-Solving Set 17

March 31st, Hilary 2021

Problem 1: *Four positive numbers p, q, r and s are in increasing order of size. One of the numbers is to be increased by 1 so that the product of the four new numbers is now as small as possible. Which number should be increased?*

(Junior Kangaroo 2021)

Problem 2: *How many positive perfect squares less than 10^6 are multiples of 24?*

(AIME 2007)

Problem 3: *How many ordered pairs of positive integers (b, c) exist where both $x^2+bx+c=0$ and $x^2+cx+b=0$ do not have distinct, real solutions?*

(AMC 12 2021)

Problem 4: *The sides of a triangle have lengths $11, 15,$ and k, where k is an integer. For how many values of k is the triangle obtuse?*

(AHSME 1995)

Problem 5: *Find all positive real solutions to the equation*

$$x + \lfloor \frac{x}{6} \rfloor = \lfloor \frac{x}{2} \rfloor + \lfloor \frac{2x}{3} \rfloor$$

(BMO1 2002)

Problem 6: *A circular hole of diameter 1 is drilled through the centre of a sphere of radius 1. What is the volume of the drilled sphere?*

(Intervarsities 2007)

Problem 7: *Find all integer solutions of*

$$8xy + 5x + 3y = 0.$$

(Intervarsities Selection Test 2009)

Problem 8: *Evaluate*

$$\int \frac{dx}{6x^5 + x}$$

(Intervarsities 2002)

Problem 9: *Find the least positive integer n for which $m^n - 1$ is divisible by 10^{2003} for all integers m with greatest common divisor $gcd(m, 10) = 1$.*

(Intervarsities 2003)

Problem 10: *Let A be a real 4×2 matrix and B be a real 2×4 matrix such that*

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Find BA.

(IMC 2004)

Problem 11: *Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,*

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

(Putnam 2009 B1)

Problem 12: *Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?*

(Putnam 2021 B1)