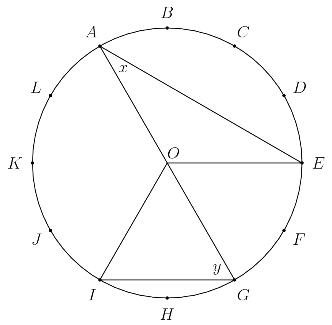
DU Mathsoc Problem-Solving Set 15

March 3rd, Hilary 2021

Problem 1: *The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y?*



(Team Maths 2022)

Problem 2: *What is the sum of the solutions to the equation $\sqrt[4]{x} = \frac{12}{7-\sqrt[4]{x}}$?*

(AIME 1986)

Problem 3: *If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?*

(AHSME 1996)

Problem 4: *Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive positive integers.*

(AIME 1987)

Problem 5: *Find all positive integer solutions of the equation $x^2 + y^2 = 2020$.*

(Intervarsities Selection Test 2019)

Problem 6: *A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and*

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \ n = 0, 1, 2, \ldots$$

Find x_{1998}.

(IRMO 1998)

Problem 7: *Ares multiplies two integers which differ by 9. Grace multiplies two integers which differ by 6. They obtain the same product T. Determine all possible values of T.*

(BMO1 2018/19)

Problem 8: *Suppose that a polynomial $p(x)$ of degree at most $n-1$ satisfies the constraints*

$$\int_0^1 x^k p(x)\,dx = 1 \text{ for all } \ k = 0, 1, \ldots, n-1.$$

Prove that $p(1) = n^2$.

(Intervarsities 2019)

Problem 9: *Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B and C, which are vertices of an equilateral triangle, and find its area.*

(Putnam 2006 B1)

Problem 10: *Is the set of positive integers n such that $n! + 1$ divides $(2012n)!$ finite or infinite?*

(IMC 2012)

Problem 11: *Let S be a set of rational numbers such that*

- *(a)* $0 \in S$;
- *(b) If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$; and*
- *(c) If $x \in S$ and $x \notin \{0, 1\}$, then $\frac{1}{x(x-1)} \in S$.*

Must S contain all rational numbers?

(Putnam 2009 A4)