DU Mathsoc Problem-Solving Set 14

February 24th, Hilary 2021

Problem 1: *Find the term independent of x in the binomial expansion of $(x - \frac{1}{x})^9$*

(Team Maths 2022)

Problem 2: *How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?*

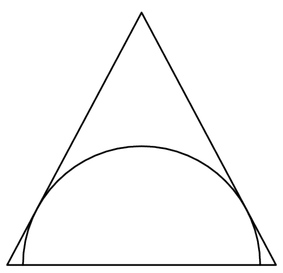
(AMC 8 2018)

Problem 3: *Find the value of*

$$\frac{1^4 + 2017^4 + 2018^4}{1^2 + 2017^2 + 2018^2}$$

(AIME 2005)

Problem 4: *A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?*



(AMC 2016)

Problem 5: *Find all solutions of the equation*

$$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2.$$

(STEP 1 1997)

Problem 6: *How many of the first 1000 positive integers can be expressed in the form*

$$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$$

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

(AIME 1985)

Problem 7: *Two points A, B are chosen randomly on the unit sphere $x^2 + y^2 + z^2 = 1$. Find the probability that the angle $\angle AOB$ is less than $60°$ when O is the origin in $\mathbb{R}^3$.*

(Intervarsities Selection Test 2019)

Problem 8: *Let A be a subset of the irrational numbers such that the sum of any two distinct elements of it be a rational number. Prove that A has two elements at most.*

(IMS 2014)

Problem 9: *Consider all 26^{26} words of length 26 in the Latin alphabet. Define the weight of a word as $\frac{1}{k+1}$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.*

(IMC 2015)

Problem 10: *The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.*

(Putnam 2000 A3)

Problem 11: *Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?*

(Putnam 2001 A1)