DU Mathsoc Problem-Solving Set 13

February 17th, Hilary 2021

Problem 1: *What is the value of the product*

$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

(AMC 8 2019)

Problem 2: *The n-th Triangular number* $T_n = 1+2+3+\cdots+n$ *is the sum of the first n natural numbers. Some of these triangles are squares (or Tri-squares), e.g.* $T_1 = 1$, $T_8 = 36 = 6^2, T_{49} = 1225 = (35)^2$.
Find the next tri-square number.

(Submission by Dami Makinde)

Problem 3: *A game uses a deck of n different cards, where n is an integer and* $n \geq 6$. *The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.*

(AIME 2005)

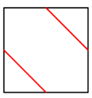
Problem 4: *Show that the product of four consecutive positive integers cannot be expressed as a product of two consecutive positive integers.*

Problem 5: *Prove that* 2005^2 *can be written in at least 4 ways as the sum of 2 perfect (non-zero) squares.*

(Flanders Junior Olympiad 2005)

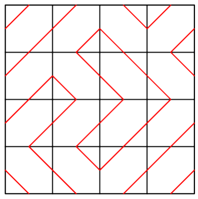
Problem 6: *Let* $f(x) = x^3 - 3x + 1$. *How many real roots does the equation* $f(f(x)) = 0$ *have?*

Problem 7: *Let N be a positive integer. A collection of* $4N^2$ *unit tiles with two segments drawn on them as shown is assembled into a* $2N \times 2N$ *board. Tiles can be rotated.*



The segments on the tiles define paths on the board. Determine the least possible number and the largest possible number of such paths.

For example, there are 9 paths on the 4×4 *board shown below.*



(Benelux 2020)

Problem 8: *A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point* $(2021, 2021)$?

(Putnam 2021 A1)

Problem 9: *Let A be a real* $n \times n$ *matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that* $2021B = A^m + B^2$.
Prove that $|detA| \leq 1$.

(IMC 2021)

Problem 10: *Show that the Diophantine equation*

$$x^2 + y^2 + 1 = xyz$$

has no solution unless $z = 3$ *(not expected to find the solutions but it is possible).*

Hint: Look up Fermat's Method of Infinite Descent

(Submission by Dami Makinde)