DU Mathsoc Problem-Solving Set 12

February 10th, Hilary Term 2022

Problem 1: *Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?*

(AMC 8 2020)

Problem 2: *Playing soccer with 3 goes as follows: 2 field players try to make a goal past the goalkeeper, the one who scores goes in goals for next game, etc.*
Alice, Bob and Cauchy played this game. Later, they tell their math teacher that A stood 12 times on the field, B 21 times on the field, C 8 times in the goal. Who scored the 6th goal?

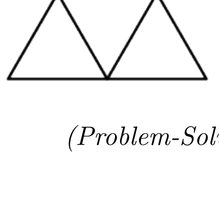
(Flanders Junior Olympiad 2003)

Problem 3: *The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.*

(AIME 2011)

Problem 4: *There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color.*

Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles may be formed?



(Problem-Solving Strategies)

Problem 5: *The area and the perimeter of the triangle with sides* $10, 8, 6$ *are equal. Find all the triangles with integral sides whose area and perimeter are equal.*

(BMO TST 2011)

Problem 6: *Determine all pairs of prime numbers* (p, q)*, with* $2 \leq p,\ q < 100$*, such that* $p+6, p+10, q+4, q+10$ *and* $p+q+1$ *are all prime numbers.*

(IRMO 2004)

Problem 7: $m < n$ *are positive integers. Let* $p = \frac{n^2+m^2}{\sqrt{n^2-m^2}}$.
(a) Find three pairs of positive integers (m, n) *that make* p *prime.*
(b) If p *is prime, then show that* $p \equiv 1 \pmod 8$.

(Turkish Junior National Olympiad 2011)

Problem 8: *Find all functions* f *from the positive integers to the positive integers such that for all* x, y *we have:*

$$2yf(f(x^2)+x) = f(x+1)f(2xy).$$

(BMO2 2022)

Problem 9: *Let* k *be a positive integer. Suppose that the integers* $1, 2, 3, \ldots, 3k+1$ *are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by* 3*? Your answer should be in closed form, but may include factorials.*

(Putnam A3 2007)

Problem 10: *For a prime number* p*, let* $GL_2(\mathbb{Z}/p\mathbb{Z})$ *be the group of invertible* 2×2 *matrices of residues modulo* p*, and let* S_p *be the symmetric group (the group of all permutations) on* p *elements.*
Show that there is no injective group homomorphism $\phi : GL_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(IMC 2021)