DU Mathsoc Problem-Solving Set 11

February 3rd, Hilary Term 2022

Problem 1: *Numbers a, b, c are such that $3a + 4b = 3c$ and $4a - 3b = 4c$. Show that $a^2 + b^2 = c^2$*

(*Polish JMO 2018*)

Problem 2: *Prove that the number of subsets of $\{1, 2, \ldots n\}$ that have an odd number of elements is 2^{n-1}.*

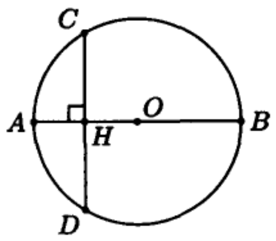
(*A Walk Through Combinatorics*)

Problem 3: *Simplify:*

$$\frac{12}{3+\sqrt{5}+2\sqrt{2}}.$$

(*Team Maths 2018*)

Problem 4: *Diameter AB of a circle has length a 2-digit integer (base ten). Reversing the digits gives the length of the perpendicular chord CD. The distance from their intersection point H to the center O is a positive rational number. Determine the length of AB.*



(*AIME 1983*)

Problem 5: *Find all positive integer N which has not less than 4 positive divisors, such that the sum of squares of the 4 smallest positive divisors of N is equal to N.*

(*Korea Junior Math Olympiad*)

Problem 6: *Let x, y, and z be positive real numbers such that $xy + yz + zx = 27$. Prove that*

$$x + y + z \geq \sqrt{3xyz}$$

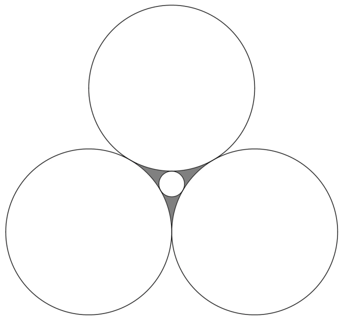
When does equality hold?

(*TST Macedonia 2020*)

Problem 7: *Eliza has a large collection of $a \times a$ and $b \times b$ tiles where a and b are positive integers. She arranges some of these tiles, without overlaps, to form a square of side length n. Prove that she can cover another square of side length n using only one of her two types of tile.*

(*BMO2 2021*)

Problem 8: *A stained glass window is to be constructed in the cathedral of Blobbyland. Part of the design includes 3 tangential circles of radius 1, with a smaller circle tangent to each of them. Find the area of the shaded region.*



(*Submission by Shea Bubendorfer*)

Problem 9: *We say that a positive real number d is good if there exists an infinite sequence $a_1, a_2, a_3, \cdots \in (0, d)$ such that for each n, the points $a_1, \ldots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find*

$$\sup\{d \mid d \text{ is good}\}.$$

(*IMC 2021*)

Problem 10: *Let S be the set of all 2×2 real matrices*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

(*Putnam 2015 B3*)