DU Mathsoc Problem-Solving Week 1

September 29th, Michaelmas 2021

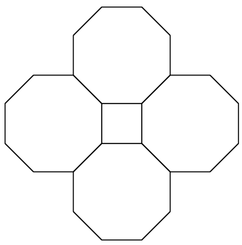
Problem 1: *Find all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers.*

(AMC 10B 2019)

Problem 2: *The product $8000 \times K$ is a square, where K is a positive integer. What is the smallest possible value of K?*

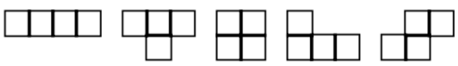
(JMO 2019)

Problem 3: *A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. (Shown here for $m = 4, n = 8$.) If $m = 10$, what is the value of n?*



(AIME 2015)

Problem 4: *Is it possible to form a rectangle with the five tetrominoes shown below?*



(Problem-Solving Strategies)

Problem 5: *Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.*

(BMO1 2017)

Problem 6: *Show that given 13 points in the plane with integer coordinates, there are three of them whose center of gravity has integer coordinates.*

(Problem-Solving Methods in Combinatorics)

Problem 7: *Show that there is a positive number in the Fibonacci sequence that is divisible by* 1000.

(Bonus: show that for any $n \in \mathbb{N}$ there is a Fibonacci number that is divisble by n)

(IRMO 1999)

Problem 8: *Let p and q respectively be the smallest and largest prime factors of n. Find all positive integers n such that $p^2 + q^2 = n + 9$.*

(Maclaurin Olympiad 2020)

Problem 9: *One of the cross sections in a rectangular box is a regular hexagon. Prove that the box is a cube.*

(Problem-Solving Strategies, Ch.12)

Problem 10: *Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.*

(Putnam 2001 A1)

Problem 11: *Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots, a_n$ of the sequence $1, 2, \ldots, n$ satisfying*

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n.$$

(IMO Shortlist 2020)

Problem 12: *If a, b, c, d are positive real numbers, show that:*

$$16 \leq (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

(Submission by Ben MacDowall)