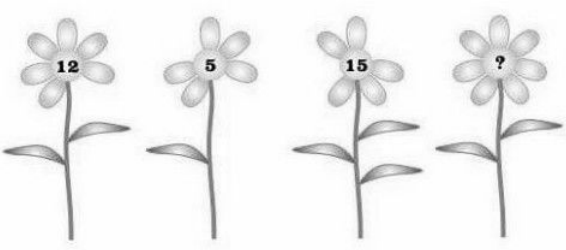
DU Mathsoc Problem Solving

Problem Set 8, Hilary 2020-21

P1. Give a real number x that is between 2 and 3 (with proof). Best number wins.

P2.



(User TuZo, AoPS Forum)

P3. Find all triples (a, b, c) of real numbers such that the following system holds:

$$\begin{cases} a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ a^2+b^2+c^2 = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \end{cases}$$

(Junior Balkan Olympiad 2020, P.1)

P4. Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

(AIME 2004, I, P.7)

P5. Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

(AIME 2005, I, P.15)

P6. Find all positive real numbers α such that there exists an infinite sequence of positive real numbers $x_1, x_2, ...$, such that

$$x_{n+2} = \sqrt{\alpha x_{n+1} - x_n}$$

for all $n \geq 1$

(Belarus 2017, P.1)

P7. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying the following equation

$$f(x + f(xy)) = xf(1 + f(y))$$

for all positive x and y

(Belarus 2017, P.3)

P8. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

(IMC 2007, P.1)

P9. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

(Putnam 2006, A2)

P10. Let $A \in \mathbb{R}^{n\times n}$ such that $3A^3 = A^2+A+I$. Show that the sequence A^k converges to an idempotent matrix. (idempotent: $B^2 = B$)

(IMC 2003, P.3)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: *Darragh Glynn, quizmaster@mathsoc.com*