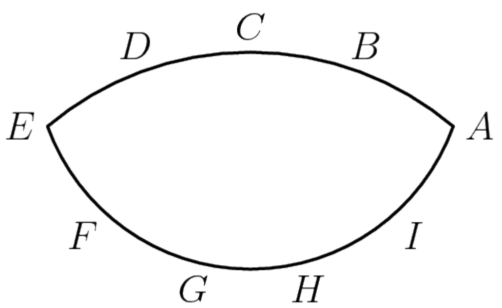
DU Mathsoc Problem Solving

Problem Set 7, Hilary 2020-21

P1. I am an odd number. Take away a letter and I become even. What number am I?

P2. Alice has as many brothers as sisters, but each brother has only half as many brothers as sisters. How many brothers and sisters are there in the family?

P3. Points A, B, C, D, and E are equally spaced on a minor arc of a circle. Points E, F, G, H, I and A are equally spaced on a minor arc of a second circle with center C as shown in the figure below. The angle $\angle ABD$ exceeds $\angle AHG$ by 12°. Find the degree measure of $\angle BAG$.



(AIME 2015 I, P.6)

P4. Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people so that exactly one person receives the type of meal ordered by that person.

(AIME 2012 P.3)

P5. Let $a > 0$ and define the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_0 = \sqrt{a}, \quad a_{n+1} = \sqrt{a + a_n}.$$

Find $\lim_{n\to\infty} a_n$.

(Problem Solving Strategies, Ch. 9)

P6. The sides and vertices of a pentagon are labelled with the numbers 1 through 10 so that the sum of the numbers on every side is the same. What is the smallest possible value of this sum?

(Flanders 2015, P1)

P7. For each integer $n \geq 2$, find all integer solutions of the following system of equations:

$$\begin{aligned} x_1 &= (x_2 + x_3 + x_4 + \cdots + x_n)^{2018} \\ x_2 &= (x_1 + x_3 + x_4 + \cdots + x_n)^{2018} \\ &\vdots \\ x_n &= (x_1 + x_2 + x_3 + \cdots + x_{n-1})^{2018} \end{aligned}$$

(Ibero-American 2018, P1)

P8. Does there exist a bijective map $f : \mathbb{N} \to \mathbb{N}$ so that $\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ is finite?

(IMC 1999, P2)

P9. Show that, for any positive integer n,

$$\sum_{r=0}^{[(n-1)/2]} \left\{ \frac{n-2r}{n} \binom{n}{r} \right\}^2 = \frac{1}{n}\binom{2n-2}{n-1},$$

where $[x]$ means the greatest integer not exceeding x, and $\binom{n}{r}$ is the binomial coefficient "n choose r", with the convention $\binom{n}{0} = 1$.

(Putnam 1965, A2)

P10. Find all polynom with coeffs a permutation of $[1, ..., n]$ and all roots rational

(IMC 2005, P4)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com