DU Mathsoc Problem Solving

Problem Set 6, Hilary 2020-21

P1. At a party, everyone shook hands with everybody else. There were 66 handshakes. How many people were at the party?

P2. It's dark. Alice the ant has ten grey socks and ten blue socks that she wants to put into pairs. All socks are exactly the same except for their colour. How many socks would she need to take to ensure she has at least one pair?

P3. Find the number of positive integers less than or equal to 2017 whose base-three representation contains no digit equal to 0.

(AIME 2017)

P4. The polynomial

$$f(x) = x^4 + ax^3 + bx^2 + cx + d$$

has real coefficients, and

$$f(2i) = f(2+i) = 0.$$

What is $a + b + c + d$?

(2007 AHSME, P.18)

P5. Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

(AHSME 2007, P.25)

P6. Find all prime numbers such that the square of the prime number can be written as the sum of cubes of two positive integers.

(Bangladesh MO 2019, P.1)

P7. Prove that,if a, b, c are positive real numbers,

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{2}{a} + \frac{2}{b} - \frac{2}{c}$$

(Bangladesh MO 2019, P.2)

P8. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that $\det(A) = \det(B)$.

(IMC 2015, P.1)

P9. Evaluate

$$\int_0^a \int_0^b e^{max\{b^2x^2, a^2y^2\}} dydx$$

where a and b are positive.

(Putnam 1989, A2)

P10. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into the square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stones should stay within the board). Sisyphus' aim is to move all n stones to square n. Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \ldots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the lest integer not smaller than x.)

(IMO Shortlist 2018, C.3)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com