DU Mathsoc Problem Solving

Problem Set 5, Hilary 2020-21

P1. A farmer says: "I only ever keep sheep, goats, and horses. In fact, at the moment they are all sheep bar three, all goats bar four, and all horses bar five." How many do they have of each animal?

(Reader's Digest)

P2. What is the minimum number of digits to the right of the decimal point needed to express the fraction $\dfrac{123\,456\,789}{2^{26}\cdot 5^4}$ as a decimal?

(A) 4 **(B)** 22 **(C)** 26 **(D)** 30 **(E)** 104

(AMC 12, P.15)

P3. Find the last three digits of the product of the positive roots of

$$\sqrt{1995}x^{\log_{1995} x} = x^2.$$

(AIME 1995 I, P.2)

P4. There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of a the value of b is unique. What is this value of b?

(AHSME 2016, P.24)

P5. Alice the ant and Bob have fallen in love (via the internet) and Alice the ant wishes to mail Bob a ring. Unfortunately, they live in the country of Kleptopia where anything sent through the mail will be stolen unless it is enclosed in a padlocked box. Alice the ant and Bob each have plenty of padlocks, but none to which the other has a key. How can Alice the ant get the ring safely into Bob's hands?

(Peter Winkler)

P6. Three different non-zero real numbers a, b, c satisfy the equations

$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p,$$

where p is a real number. Prove that $abc + 2p = 0$.

(IrMO 2014, P.4)

P7. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

(AIME 1995 1, P.6)

P8. Let $d_1, d_2, \dots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

(Putnam 2012, A1)

P9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - f(y) \in \mathbb{Q} \quad \text{for all} \quad x - y \in \mathbb{Q}$$

(IMC 2008 I, P.1)

P10. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

(IMC 2008, II, P.2)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com