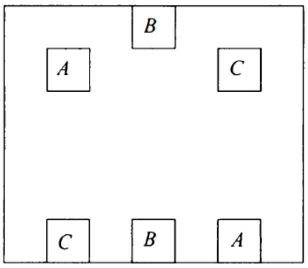
DU Mathsoc Problem Solving

Problem Set 4, Hilary 2020-21

P1. Can you connect each small box on the top with its same-letter mate on the bottom with paths that do not cross one another nor leave the boundaries of the large box?



(Zeitz)

P2. Alice the ant has a bottle containing a quart of coffee, and Ben has a bottle containing a quart of milk. Ben pours a small amount of milk into Alice's bottle, and Alice then pours back into Ben's bottle until both bottles contain a quart of liquid. What is the relationship between the fraction of milk in Alice's bottle and the fraction of coffee in Ben's bottle?

(Zeitz)

P3. The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

(AMC 12, 2020, P.5)

P4. Which of the following numbers is a perfect square?

$$\frac{14!15!}{2}, \quad \frac{15!16!}{2}, \quad \frac{16!17!}{2}, \quad \frac{17!18!}{2}, \quad \frac{18!19!}{2}$$

(AMC 10 2014, P.8)

P5. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

(AIME 2018 II, P.2)

P6. For some integer m, the polynomial

$$x^3 - 2011x + m$$

has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

(AIME 2011 I, P.15)

P7. Find all polynomial solutions of the functional equation

$$f(x)f(x+1) = f(x^2 + x + 1).$$

(Problem Solving Strategies, Ch. 10)

P8. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

(Putnam 2012, A1)

P9. Determine all complex numbers λ for which there exists a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(IMC 2017, P.1)

P10. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geqslant 1 \quad \text{for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)dx \geqslant \frac{1}{3}.$$

(IMC 2019, P.3)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com