

DU Mathsoc Problem Solving

Problem Set 9, Michaelmas 2020

P1. Consider the integer

$$N = 9 + 99 + 999 + 9999 + \dots + \underbrace{99\dots 99}_{321 \text{ digits}}.$$

Find the sum of the digits of N .

(AIME 2019)

P2. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \dots$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

(AIME 2016)

P3. Let \mathcal{S} be a set of n different points in the plane so that no three lie on a line. Show that there are at least $\frac{n(n-2)}{3}$ triangles whose vertices are among the points of \mathcal{S} and such that the triangles do not contain any other points of \mathcal{S} .

(Donal Hurley)

P4. What is the maximum value of $\frac{(2^t-3t)t}{4^t}$ for real values of t ?

(AHSME 2020, P.22)

P5. Suppose a, b, c are positive real numbers. Prove that

$$abc(a + b + c) \leq a^4 + b^4 + c^4,$$

with equality if and only if $a = b = c$.

(Finbarr Holland)

P6. Can you arrange the numbers $1, 2, 3, \dots, 9$ along a circle so that the sum of any two neighbours is not divisible by $3, 5$ or 7 ?

(Problem Solving Strategies, Ch. 5)

P7. Let a_1, a_2, \dots, a_n, k , and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \dots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\dots(x+a_n)$$

has no positive roots.

(IMO Shortlist 2017, A.1)

P8. Given that $\{x_1, x_2, \dots, x_n\} = \{1, 2, \dots, n\}$, find, with proof, the largest possible value (as a function of n , with $n \geq 2$), of

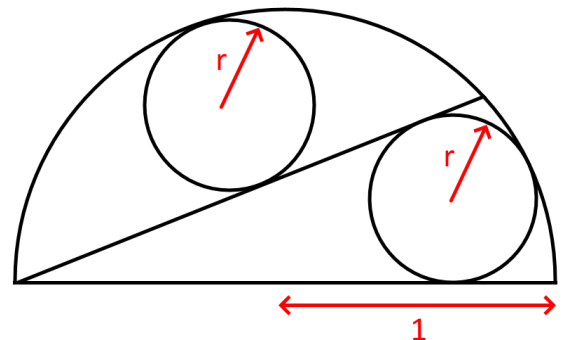
$$x_1 x_2 + x_2 x_3 + \dots + x_{n-1} x_n + x_n x_1.$$

(Putnam 1996, B3)

P9. Let $f : \{1, 2, 3, \dots\} \rightarrow \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f .

(IMO Shortlist 2018, N.6)

P10. The two circles in the diagram below have common radius r . Find the value of r .



(Submission by Lucy Deacon)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com