DU Mathsoc Problem Solving

Problem Set 8, Michaelmas 2020

P1. How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

(AHSME 2020, P.8)

P2. Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

(AHSME 2014, P.14)

P3. Let x be a real number such that

$$\sin^{10} x + \cos^{10} x = \tfrac{11}{36}.$$

Then

$$\sin^{12} x + \cos^{12} x = \tfrac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

(AIME 2019)

P4. There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

(AIME 2020)

P5. Does there exist an even positive integer n for which $n + 1$ is divisible by 5 and the two numbers $2^n + n$ and $2^n - 1$ are co-prime?

(IrMO 2017, P.5)

P6. Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

(IMO Shortlist 2012, N.2)

P7. Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

(Putnam 1996, A1)

P8. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers a, b, c with $a + b + c = 0$, the following equality holds:

$$\begin{aligned} &f(a)^2 + f(b)^2 + f(c)^2 \\ &\quad = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a). \end{aligned}$$

(IMO 2012, Problem 4)

P9. Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

(Putnam 1985, B3)

P10. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

(Putnam 2000, A6)

Open Problem. Is there a number (other than 1) that appears more than 8 times in Pascal's Triangle? (Perhaps 1000000000000066600000000000001? Or the least-odd prime, 2?)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com