DU Mathsoc Problem Solving

Problem Set 7, Michaelmas 2020

P1. How many different real numbers x satisfy the equation $(x^2-5)^2 = 16$?

(2019 AMC 8)

P2. Bob and Janice each choose a real number in the interval $[0, 6]$. What is the probability that the sum of the squares of their numbers is less than 48?

(AoPS, StarryNight7210)

P3. The sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$$

is defined by letting $a_1 = 2$, $a_{n+1} = 2^{a_n}$. What is the first term in this sequence greater than 1000^{1000}?

(AMC Senior Practise)

P4. Find $a^5 + b^5 + c^5$ if

$$\begin{aligned} a + b + c &= 1, \\ a^2 + b^2 + c^2 &= 2, \\ a^3 + b^3 + c^3 &= 3. \end{aligned}$$

(Email Submission)

P5. A rational number written in base eight is $\underline{ab}.\underline{cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb}.\underline{ba}$. Find the base-ten number $\underline{abc}$.

(AIME 2017)

P6. A set contains four numbers. The six pairwise sums of distinct elements of the set, in no particular order, are 189, 320, 287, 234, x, and y. Find the greatest possible value of $x + y$.

(AIME 2017)

P7. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1),$$

and

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(Putnam 1996, B2)

P8. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

(Putnam 1989, A1)

P9. a, b, c are positive reals with product 1. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

(IMO 2000, Problem 2)

P10. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$ and replace them with $2x$ and $y - x$.

(Putnam 1993, B6)

Prove without words that

$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com