DU Mathsoc Problem Solving

Problem Set 6, Michaelmas 2020

P1. Given that $n-4$ is divisible by 5, which of the following numbers are also divisible by 5?

$$n^2-1, n^2-4, n^2-16, n+4, n^4-1$$

(Mandelbrot Competition #3)

P2. What is the units digit of $7^{(7^7)}$?

(MAΘ 1990)

P3. Let $q(x)$ be a polynomial with real coefficients. Find all polynomials $p(x)$ such that

$$p - p' + p'' - p''' + \ldots = q.$$

(Submission by Mark Heavey)

P4. Find $f(2)$, given that f is a function satisfying

$$f(x) + 2f(1/(1-x)) = x.$$

(MAΘ 1992)

P5. Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $q(x)$ defined by

$$q(x) = p(x+n) - p(x)$$

has integer coefficients.

(Stephen Buckley)

P6. For any positive integer n define

$$E(n) = n(n+1)(2n+1)(3n+1)\ldots(10n+1).$$

Find the greatest common divisor of $E(1), E(2), E(3), \ldots, E(2009)$.

(Marius Ghergu)

P7. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

(AIME 2020)

P8. Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labeled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The *value* of a square is the sum of its row number and column number. The *score* of an arrangement of rooks is the least value of any occupied square.The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

(AIME 2016)

P9. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(IMO Shortlist 2018, N.1)

P10. Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

(Putnam 1993, B3)

Jensen's Inequality Let $\omega_1, \ldots, \omega_n$ be nonnegative reals with $\omega_1 + \ldots + \omega_n = 1$, and let f be a function which is convex on the interval I. Then

$$f(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_n x_n) \\ \leq \omega_1 f(x_1) + \omega_2 f(x_2) + \ldots + \omega_n f(x_n),$$

for any $x_1, \ldots, x_n \in I$.

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com