DU Mathsoc Problem Solving

Problem Set 5, Michaelmas 2020

P1. Show that for all prime numbers p greater than 3, 24 divides $p^2 - 1$.

(AHSME 1973)

P2. Prove the following equalities:

$$\sin 10^\circ \sin 20^\circ \sin 30^\circ = \sin 10^\circ \sin 10^\circ \sin 100^\circ,$$

$$\sin 20^\circ \sin 20^\circ \sin 30^\circ = \sin 10^\circ \sin 20^\circ \sin 80^\circ,$$

$$\sin 20^\circ \sin 30^\circ \sin 30^\circ = \sin 10^\circ \sin 40^\circ \sin 50^\circ.$$

(M&IQ 1992)

P3. Let P be a point chosen uniformly at random in the interior of the unit square with vertices at $(0,0),(1,0),(1,1)$, and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8},\frac{3}{8}\right)$ is greater than or equal to $\frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

(AIME 2020)

P4. Find the number of 7-tuples of positive integers (a,b,c,d,e,f,g) that satisfy the following system of equations:

$$abc = 70, \quad cde = 71, \quad efg = 72.$$

(AIME 2019)

P5. How many subsets of $\{1,2,3,\ldots,n\}$ have no two consecutive numbers?

(Problem Solving Strategies, Ch. 5)

P6. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

(Putnam 2008, B1)

P7. Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbours on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < \frac{2}{n}$.

(Putnam 2003, B2)

P8. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \ldots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \ldots (1+a_n)^n > n^n.$$

(IMO 2012, Problem 2)

P9. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd?

(Putnam 2011, A4)

P10. Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0,1,2,\ldots,p-1\}$,

$$\sum_{k=0}^{p-1} k! n^k$$

is not divisible by p.

(Putnam 2011, B6)

"Problems worthy of attack
Prove their worth by fighting back"

– Piet Hein

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com