DU Mathsoc Problem Solving

Problem Set 11, Michaelmas 2020

P1. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

(AIME 2017 P.1)

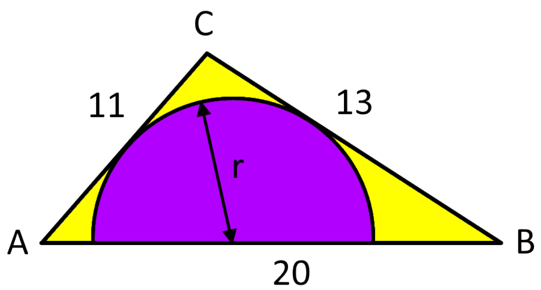
P2. At a cinema, the manager announces that a free ticket will be given to the first person in line whose birthday is the same as someone in line who has already bought a ticket. You have the option of getting in line at any time. Assuming that you don't know anyone else's birthday, and that birthdays are uniformly distributed throughout a 365-day year, what position in line gives you the best chance of being the first duplicate birthday?

(Nick's Mathematical Puzzles)

P3. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

(AIME 2020 P.12)

P4. In $\triangle ABC$, side $AB = 20$, $AC = 11$, and $BC = 13$. Find the diameter of the semicircle inscribed in $\triangle ABC$, whose diameter lies on AB, and that is tangent to AC and BC.



(Nick's Mathematical Puzzles)

P5. Find a function $f : \mathbb{N} \to \mathbb{N}$ (with $f(n) \geq n$) so that

$$\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{f(n)} = \infty$$

(Darragh Glynn)

P6. Show that if

(a) $a_1, a_2, \ldots, a_n$ are integers, then some consecutive subsequence $a_k, a_{k+1}, \ldots, a_m$ has a sum that is divisible by n.

(b) $\{a_1, a_2, \ldots, a_{2p-1}\}$ is a set of integers, p prime, then there is a subset of size p whose sum is divisible by p.

(Stan Wagon)

P7. Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices over the complex numbers. Show that if T is a linear map $M_n(\mathbb{C}) \to \mathbb{C}$ such that $T(AB) = T(BA)$ for all A, B, then there is a scalar $c \in \mathbb{C}$ such that $T = c \cdot \mathrm{tr}$, where tr denotes the trace operator.

(Submission by James Murphy)

P8. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2 \left\{ \frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n) \right\} dx_1 dx_2 \cdots dx_n.$$

(Putnam 1965, B1)

P9. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(IMO Shortlist 2018, N.1)

P10. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

(Putnam 1988, A2)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com