DU Mathsoc Problem Solving

Problem Set 10, Michaelmas 2020

P1. The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

(AIME 2020)

P2. Show that if m divides $(m-1)! + 1$ then m is prime.

(Problem Solving Strategies, Ch. 6)

P3. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x, y)) = 60$$

$$\log_{10} y + 2\log_{10}(\mathrm{lcm}(x, y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

(AIME 2019)

P4. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

(AIME 2020)

P5. Let $n = (p^2 + 2)^2 - 9(p^2 - 7)$ where p is a prime number. What is the smallest value of the sum of the digits of n and for what prime numbers p is this value attained?

(Gordon Lessels)

P6. There are 650 points inside a circle of radius 16. Prove that there exists a ring with inner radius 2 and outer radius 3 covering ten of these points.

(Problem Solving Strategies, Ch.4)

P7. Let $n \geq 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of the subsets of cardinality m.

(IMO Shortlist 2018, C.1)

P8. Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

(Putnam 2008, B2)

P9. A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

(IMO 2016, P.4)

P10. In how many ways can the integers from 1 to n be ordered subject to the condition that, except for the first integer on the left, every integer differs by 1 from some integer to the left of it?

(Putnam 1965, A5)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com