DU Mathsoc Problem Solving

Problem Set 4, Michaelmas 2020

P1. Six points are placed in a 3×4 rectangle. Show that two of these points are a distance $\leq \sqrt{5}$ from each other.

(Problem Solving Strategies, Ch.4)

P2. How many positive integers less than 101 are multiples of 5 or 7, but not both?

(Mandelbrot Competition #1)

P3. Alice and Bob alternately draw diagonals of a regular 1988-gon. They may connect two vertices if the diagonal does not intersect an earlier one. The loser is the one who cannot move. If A starts, which player has a winning strategy?

(Problem Solving Strategies, Ch.13)

P4. There are several circles of total circumference length 10 inside a square of side length 1. Show that there exists a straight line which intersects at least four of these circles.

(Problem Solving Strategies, Ch.4)

P5. Find the least odd prime factor of

$$2019^8 + 1.$$

(AIME 2019)

P6. Find the sum of all positive integers n such that $\sqrt{n^2 + 85n + 2017}$ is an integer.

(AIME 2017)

P7. Show that the polynomial

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

has no multiple roots.

(Problem Solving Strategies, Ch. 6)

P8. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$.

(IMO Shortlist 2018, A.1)

P9. The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y) K(x, y)\, dy = g(x)$$

and

$$\int_0^1 g(y) K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

(Putnam 1993, B4)

P10. Show that for any positive integer k, there exists a sequence of consecutive positive integers of length k such that no member of the sequence is the power of a prime. (Bonus: give a one-sentence proof.)

(Submission by Mark Heavey)

Finally, some inspiration for the week ahead:

"If you improve 1% every day, you will have improved 365% in one year"

– someone wise, probably

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com