DU Mathsoc Problem Solving

Problem Set 3, Michaelmas 2020

P1. If $f(x) = \frac{4}{x-1}$ and $g(x) = 2x$, then find all x such that $f(g(x)) = g(f(x))$.

(MAΘ 1991)

P2. Find the number of ordered pairs of positive integers (m, n) such that $m^2 n = 20^{20}$.

(AIME 2020)

P3. Find the value of

$$\sin^2 10^\circ + \sin^2 20^\circ + \sin^2 30^\circ + \dots + \sin^2 80^\circ + \sin^2 90^\circ.$$

(MAΘ 1992)

P4. Prove that if n is a positive integer either $3n$ or $7n$ contains an odd digit.

(Gordon Lessels)

P5. Given an n-tuple of numbers $(x_1, \dots, x_n)$ where each $x_i = +1$ or -1, form a new n-tuple

$$(x_1x_2, x_2x_3, x_3x_4, \dots, x_nx_1),$$

and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the n-tuple

$$(1, 1, 1, \dots, 1).$$

(Donal Hurley)

P6. Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}.$$

(Putnam 1999, A4)

P7. For the polynomial

$$P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$$

define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

(AIME 2016)

P8. Find the last three digits of 7^{9999}.

(Problem Solving Strategies, Ch. 6)

P9. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

(Putnam 1993, B5)

P10. Let a and b be integers with $a > b > 1$. Let $\lambda(a, b)$ denote the number of individual applications of the division algorithm required by Euclid's algorithm to compute the greatest common divisor of a and b. Find constants c and d such that

$$\lambda(a, b) \leq c \log b + d.$$

(Cambridge Part II, Number Theory)

Prove without words that for $\alpha > 0$,

$$\int_0^1 \left(x^\alpha + x^{1/\alpha}\right) dx = 1$$

(Art of Problem Solving)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com