DU Mathsoc Problem Solving

12th of October, Michaelmas 2020

P1. Express in simplest form:

$$\prod_{k=2}^{n}\left(1+\frac{1}{k}\right)$$

P2. Mike starts at one corner of a tetrahedron. In a *move*, he walks along any edge to a different corner. In how many ways can he end up where he started after 6 moves?

(Art of Problem Solving Vol. 2)

P3. If $f(z)=\frac{z+1}{z-1}$, then find $f^{1991}(2+i)$.

(MAΘ 1991)

P4. Let x, y, and z be real numbers satisfying the system

$$\log_2(xyz-3+\log_5 x)=5,$$

$$\log_3(xyz-3+\log_5 y)=4,$$

$$\log_4(xyz-3+\log_5 z)=4.$$

Find the value of $|\log_5 x|+|\log_5 y|+|\log_5 z|$.

(AIME 2016)

P5. For which integers n is

$$\frac{16(n^2-n-1)^2}{2n-1}$$

also an integer?

(2018 UK IMOK, M5)

P6. Find the smallest positive integer n such that for every integer m with $0<m<1993$, there exists an integer k for which

$$\frac{m}{1993}<\frac{k}{n}<\frac{m+1}{1994}.$$

(Putnam 1993, B1)

P7. Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

(HMMT 2019, Algebra and NT)

P8. A number written in base 10 is a string of 3^{2013} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

(BMO 2013, P3)

P9. Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

(Putnam 2019, A4)

P10. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

(IMO Shortlist 2017, C.1)

The 5/8 Theorem. If the probability that two randomly chosen elements of a finite group commute is greater than 5/8, then the group is abelian.

(Submission by Liam Kavanagh)

Join the Mathsoc Discord server at 6pm this Thursday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com