

DU Mathsoc Problem Solving

Problem Set 3, Hilary 2020-21

P1. One day Alice said, “I have been alive during all or part of 5 decades.” Rounded to the nearest year, what is the youngest she could have been?

(The Art and Craft of Problem Solving)

P2. At least how many times must you flip a fair coin before there is at least a 50% probability that you will get at least 3 heads?

(The Art and Craft of Problem Solving)

P3. The expressions

$$A = 1 \times 2 + 3 \times 4 + 5 \times 6 + \dots + 37 \times 38 + 39$$

$$B = 1 + 2 \times 3 + 4 \times 5 + \dots + 36 \times 37 + 38 \times 39$$

are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B .

(AIME 2015 I, P.1)

P4. There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p .

(AIME 2015 I, P.3)

P5. Show that 2005^{2005} is a sum of two perfect squares, but not a sum of two perfect cubes.

(IrMO 2005, P.1)

P6. Prove that there are no perfect squares in the array below:

11	111	1111	...
22	222	2222	...
33	333	3333	...
44	444	4444	...
55	555	5555	...
66	666	6666	...
77	777	7777	...
88	888	8888	...
99	999	9999	...

(Flanders Math Olympiad, 2002)

P7. If n is an integer and $n > 11$, show that $n^2 - 19n + 89$ is not a square.

(Problem Solving Strategies, Ch. 6)

P8. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, Alice and B . Beginning with Alice, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both Alice and B , what is the probability that Alice wins?

(Putnam 1993, B2)

P9. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3 + 3n)^2}{n^6 - 64}.$$

(IMC 2019, P.1)

P10. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

(Putnam 1985, B2)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com