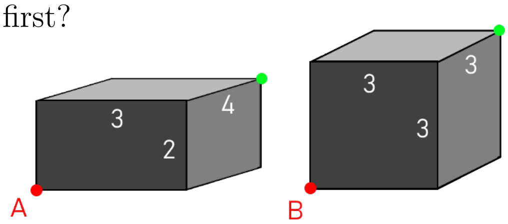
DU Mathsoc Problem Solving

Problem Set 2, Hilary 2020-21

P1. Alice the Ant is positioned on a $2 \times 3 \times 4$ cuboid, and Bob, another ant, is positioned on a $3 \times 3 \times 3$ cube, at the corners indicated. At the same time, they each begin crawling along the shortest path to the opposite corner of their respective cube, where a tasty and nutritious crumb has been placed. Both ants move at the same speed. Who makes it to their crumb first?

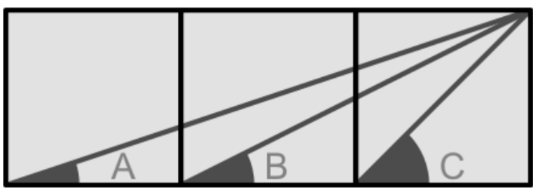


(Popular Mechanics)

P2. Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

(AIME 2012 P.1)

P3. Using only elementary geometry (not even trigonometry), prove that angle C equals the sum of angles A and B.



(Martin Gardner)

P4. Find the least positive integer N such that the set of 1000 consecutive integers beginning with $1000 \cdot N$ contains no square of an integer.

(AIME 2013, II P.6)

P5. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

(Problem Solving Strategies, Ch. 4)

P6. Each term in a sequence $1, 0, 1, 0, 1, 0, \ldots$ starting with the seventh is the sum of the last 6 terms mod 10. Prove that the sequence $\ldots, 0, 1, 0, 1, 0, 1, \ldots$ never occurs.

(Problem Solving Strategies, Ch. 1)

P7. Find all positive integers n for which both $837 + n$ and $837 - n$ are cubes of positive integers.

(IrMO 2015, P. 3)

P8. Several positive integers are written in a row. Iteratively, Alice(!) chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can only perform finitely many such iterations.

(IMO Shortlist 2012, C.1)

P9. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

(Putnam 1985, B2)

P10. Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P < \deg Q$.

(Putnam 2007, B4)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com