DU Mathsoc Problem Solving

Problem Set 1, Hilary 2020-21

P1. Alice the ant and Bob play ten rounds of rock-paper-scissors. You know that:

- Alice uses rock three times, scissors six times and paper once.
- Bob uses rock twice, scissors four times, and paper four times.
- There are no ties in all ten games.

Who wins, and by how much?

(Hubert Phillips)

P2. A farmer has four straight pieces of fencing: 1, 2, 3, and 4 yards in length. What is the maximum area he can enclose by connecting the pieces? Assume the land is flat.

(Nick's Mathematical Puzzles)

P3. Suppose you have an unfair coin which does not land on each side 50% of the time. How could you use this coin to simulate the 50/50 odds of a fair coin flip?

(John von Neumann)

P4. Simplify the product

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{a^4}\right)\cdots\left(1+\frac{1}{a^{2^{100}}}\right)$$

(The Art and Craft of Problem Solving)

P5. Show that each number in the sequence

$$49, 4489, 444889, 44448889, 4444488889, \ldots$$

is a perfect square.

(The Art and Craft of Problem Solving)

P6. A 23×23 square is completely tiled by $1 \times 1, 2 \times 2$ and 3×3 tiles. What is the minimum number of 1×1 tiles needed?

(All-Union Olympiad 1989)

P7. Every point on a sphere is coloured either red or blue. Prove that there exist three points of the same colour which are the vertices of an equilateral triangle.

(Problem Solving Strategies, Ch. 2)

P8. Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

(IMO Shortlist 2011, A3)

P9. Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

(Putnam 1985, B1)

P10. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)dx$. For which integers $m, 1 \leq m \leq 10$ is $I_m \neq 0$?

(Putnam 1985, A5)

Join the Mathsoc Discord server at 7pm this Friday for a discussion of this week's problems! Submissions, solutions and questions welcome: Darragh Glynn, quizmaster@mathsoc.com