Mathsoc Problem Solving / Puzzle League 2019/2020

Week 2'

03/02/2020

1. (Art of Problem Solving) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients and let m be a positive integer such that $(x+3)(x+7)P(x) + (x-2)(x-5)Q(x) = m$. What is the minimum value of m?

2. (UCC Maths Enrichment) The roots r_1, r_2, r_3 of the polynomial $x^3 - 2x^2 - 11x + a$ satisfy $r_1 + 2r_2 + 3r_3 = 0$. Find all possible values of a.

3. (UCD Maths Enrichment) Find the minimum value of

$$x + \frac{8}{y(x-y)}$$

for $x > y > 0$.

4. (Estonia 2002) . Juku built a robot that moves along the border of a regular octagon, passing each side in exactly 1 minute. The robot starts in some vertex A and upon reaching each vertex can either continue in the same direction, or turn around and continue in the opposite direction. In how many different ways can the robot move so that after n minutes it will be in the vertex B opposite to A?

5. (IrMO 2003) Find all solutions in integers of the equation

$$(m^2 + n)(m + n^2) = (m + n)^3$$

6. (China TST 2005) Let $a, b, c, d > 0$ and $abcd = 1$. Show that

$$\frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} + \frac{1}{(d+1)^2} \geq 1$$

7. (Berkeley Problems in Mathematics) Suppose that $f''(x) = (x^2 - 1)f(x)$ and that $f(0) = 1, f'(0) = 0$. Show that $f(x) \to 0$ as $x \to \infty$.

8. (Putnam 1952) The polynomial $p(x)$ has all integral coefficients. The leading coefficient, the constant term, and $p(1)$ are all odd. Show that $p(x)$ has no rational roots.

9. (Putnam 1996) Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

10. (Putnam 2019) Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^22 + y^2 = 2^k$ for some $k \leq n$.Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Weekly Problem:

- Find all $n \in \mathbb{N}_{>0}$ such that $x^2 + x + 1 \mid x^{2n} + x^n + 1$.
- Find all $n \in \mathbb{N}_{>0}$ such that $37 \mid$