Mathsoc Problem Solving / Puzzle League 2019/2020

Week 1'

27/01/2020

1. (2020 CCA Math Bonanza Team Round # 5) Find all real numbers (x, y) satisfying the following simultaneous equations:

$$3x^2 + 3xy + 2y^2 = 2 \tag{1}$$
$$x^2 + 2xy + 2y^2 = 1 \tag{2}$$

2. (Polish mathematical Olympiad) Let $S = \{1, 2, 3, 4, 5\}$. Find the number of functions $f \colon S \to S$ such that $f^{50}(x) = x$ for all $x \in S$.

3. (Coin problem) Let m, n be two coprime positive integers. Show that the number $mn - m - n$ is the greatest positive integer which cannot be expressed in the form $am + bn$ where a, b are positive integers.

4. (Putnam preparation) Prove that there is no positive integer n for which n^5 can be written as the product of six consecutive positive integers.

5. (IrMO 2015) A regular polygon with $n \geq 3$ sides is given. Each vertex is coloured either red, green or blue, and no two adjacent vertices of the polygon are the same colour. There is at least one vertex of each colour.
   Prove that it is possible to draw certain diagonals of the polygon in such a way that they intersect only at the vertices of the polygon and they divide the polygon into triangles so that each such triangle has vertices of three different colours.

6. (IrMO 2015) Suppose a doubly infinite sequence of real numbers $\ldots, a_{-3}, a_{-2}, a_{-1}, a+0, a_1, a_2, a_3, \ldots$ has the proeprty that

$$a_{n+3} = \frac{a_{n+2} + a_{n+1} + a_n}{3}$$

   for all $n \in \mathbb{Z}$. Show that if a_n is bounded, then a_n is a constant sequence.

7. (Berkeley Problems in Mathematics) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable periodic function of period 1, and nonnegative. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{1 + cf(x)}\right) \to 0$$

as $c \to \infty$.

8. (Putnam preparation) Let ABC be a triangle such that $\angle CAB = \angle CBA = \alpha$. Assume that the incircle of the triangle touches the sides AB and AC in C' and B' respectively. Assume that I is the incentre of $\triangle ABC$ and $AC' = x$. Find an expression, in terms of x and α, for the area of the curvilinear triangle bounded by the lines AB and AC and the arc of the incircle between C' and B'.

9. (Putnam 2019) Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, C are nonnegative integers.

10. (Putnam 2019) Let Q be an $n \times n$ real orthogonal matrix (that is, a matrix satisfying $QQ^T = I$, where I is the identity matrix), and let u be a vector of unit length. Let $P = I - 2uu^T$. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

Weekly Problem:

An elephant must travel 1000km across the Alps and French countryside in order to reach Italy. They have 3000 watermelons at the starting point, but may only carry 1000 watermelons at a time. For each kilometre walked, the elephant must eat a watermelon. What is the maximum number of watermelons that the elephant can transport to Italy?