Mathsoc Problem Solving / Puzzle League 2019/2020

Week 12

25/11/2019

1. (UCD maths enrichment) Let Q be a point inside a triangle ABC of area Δ. Three lines pass through Q and are parallel with the sides of the triangle. These lines divide the initial triangle into six parts, three of which are triangles of areas S_1, S_2 and S_3.
   Prove that $\sqrt{\Delta} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}$.

2. (IrMO 1995) Let a, b, c be complex numbers such that all three roots of the equation $z^3+az^2+bz+c = 0$ satisfy $|z| = 1$. Prove that all three roots of the equation $w^3 + |a|w^2 + |b|w + |c| = 0$ also satisfy $|w| = 1$.

3. (Eugene Gath UL) Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{2}{3}$ for all $n \geq 2$. (Only using algebraic methods and without evaluating the infinite sum).

4. (Putnam preparation) Determine all functions $\mathbb{N}\colon \to \mathbb{N}$ (not including 0) such that $f(4) = 4$ and
   $$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)}$$

5. (Bernd Kreussler, of UL MIC) Let n be a positive integer such that $n+1$ is divisible by 24. Prove that the sum of all positive divisors of n is divisible by 24.

6. (IMO 1966) In a mathematical contest, three problems, A, B, C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A.
   How many students solved only problem B?

7. (Putnam 1992) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S ?

8. (Putnam 1984) Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points which are at a distance at most 1 from some point of A (in particular, B contains every point of A). Express the volume of B as a polynomial in a, b, c.

9. (Putnam 1981) Find

$$\lim_{n\to\infty}\left[\frac{1}{n^5}\sum_{h=1}^{n}\sum_{k=1}^{n}(5h^4 - 18h^2k^2 + 5k^4)\right]$$

10. (Putnam 1980) For which real numbers a does the sequence $u_0 = a$, $u_{n+1} = 2u_n - n^2$ satisfy $u_n > 0$ for all $n \in \mathbb{N}$?

Weekly Problem:

It is noted that if you start at the north pole, walk a kilometre south, walk a kilometre east, and walk a kilometre north, you will end up back where you started! Can you find another point on the globe where you can walk a kilometre south, walk a kilometre east, and walk a kilometre north to end up where you started?