Mathsoc Problem Solving / Puzzle League 2019/2020

Week 11

18/11/2019

1. (UCD maths enrichment) There are 13 white chameleons, 15 black chameleons, and 17 red chameleons on an island. When any two chameleons of different colours meet, they both change into the third colour. Is it possible for all chameleons to eventually have the same colour?

2. Let $a_1, a_2, \ldots$ be an arithmetic sequnce of positive integers. Show that if there exists at least one element in the sequence that is the square of a positive integer, then there are infinitely many square numbers in the sequence.

3. (Eugene Gath UL) Find all real x such that

$$\sqrt{1 - \frac{1}{x}} + \sqrt{x - \frac{1}{x}} \geq \frac{x-1}{x}$$

4. (Putnam preparation) If $f\colon [0,1] \to \mathbb{R}$ is a continuous function that is differentiable on $(0,1)$, prove that there exists a point $c \in (0,1)$ such that $f(1) - f(0) = \dfrac{f'(c) = f'(c)}{2}$

5. (IrMO 2016) Do there exist 4 polynomials $P(x), Q(x), R(x)$ and $S(x)$ such that the sum of any three of them has a real root, but the sum of any two of them has no real root?

6. (Putnam preparation) Assume that 101 distinct points are place in a 10×10 square such that no three of them lie on a line. Prove that we can choose three of the given points to form a triangle whose area is at most 1.

7. (EGMO 2019) Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b$$

8. (Putnam 1994) Suppose the sequence $a_1, a_2, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum_{n \geq 1} a_n$ diverges.

9. Prove that if $11z^{10} + 10iz^9 + 10iz + 11 = 0$ then $|z| = 1$.

10. (Putnam 2000) Prove that there exists infinitely many integers n such that n, $n+1$, and $n+2$ are all the sum of two squares of integers.

Weekly Problem: A person has two children, one of whom is a boy born on a Tuesday. What is the probability that the man has two sons?