Mathsoc Problem Solving / Puzzle League 2019/2020

Week 8

04/11/2019

1. (IrMO 2017) Find all solutions to the following simultaneous equations for real numbers a, b, c: $a + b + c = 0$, $a^2 + b^2 + c^2 = 1$, $a^3 + b^3 + c^3 = 4abc$

2. A frog is jumping in the coordinate plane according to the following rules: (i) From any lattice point (a, b), the frog can jump to $(a + 1, b)$, $(a, b + 1)$, or $(a + 1, b + 1)$. (ii) There are no right angle turns in the frog's path. How many different paths can the frog take from $(0, 0)$ to $(5, 5)$?

3. (EGMO selection test 2016) Prove that for all real numbers a, b, c we have that

$$\frac{2a + b}{b + 2c} + \frac{2b + c}{c + 2a} + \frac{2c + a}{a + 2b} \geq 3$$

4. (Olympiad polynomials) Find all polynomials P such that $P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right)$

5. (IrMO 2015) In the triangle ABC, the length of the altitude from A to BC is 1. Let D be the midpoint of AC. What are the possible lengths of BD?

6. (Berkeley Problems in Mathematics) Let B denote the matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

   where a, b, c are real and $|a|, |b|$, and $|c|$ are all distinct. Show that there are exactly four symmetric matrices (matrices satisfying $A^T = A$) of the form BQ, where $Q^{-1} = Q^T = I_3$ and the determinant of Q is 1.

7. (Putnam preparation) Determine whether the series

$$\sum_{n=0}^{\infty} \frac{1}{(\ln n)^{\ln(\ln n)}}$$

   is convergent or divergent.

8. (IMO, 1980's I think) Show that there exists a sequence of $10,000$ consecutive positive integers such that none of them is the power of a prime.

9. (Putnam preparation) Let R be a ring. If for all $a, b \in R$, $(ab)^2 = a^2b^2$ prove that $ab = ba$, ie the ring is commutative.

10. (Putnam 1993) Let O be the origin. The horizontal line $y = c$ intersects the y-axis at R and intersects the curve $y = 2x - 3x^3$ in the first quadrant at P and Q. Find c so that the area OPR bounded by the $y - axis$, the curve and the line $y = c$, is equal to the area under the curve and above the line $y = c$.

Weekly Problem: A ration pudding is to be divided up among 50 peasants in a particular noniuniform manner. The first peasant gets $1/50$ of the pudding, the second peasant gets $2/50$ of the remaining pudding, the third gets $3/50$ of what remains, and so on, in order, until the final 50th peasant gets $50/50$ of what remains. Which peasant gets the largest slice of pudding?