

Mathsoc Problem Solving / Puzzle League 2019/2020

Week 5

14/10/2019

1. (UCD Maths Enrichment) Prove the following inequality, for $x \geq -1$ a real number and n a positive integer:

$$(1 + x)^n \geq 1 + nx$$

2. Determine whether the number $\alpha = \sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$ is rational or irrational.
3. (UCC Maths Enrichment) Find all right-angled triangles with positive integer side lengths whose perimeter is equal to its area.
4. (Putnam preparation) If a_1, \dots, a_n are positive real numbers, prove that

$$\frac{a_1}{a_2 + \dots + a_n} + \frac{a_2}{a_1 + a_3 + \dots + a_n} + \dots + \frac{a_n}{a_1 + \dots + a_{n-1}} \geq \frac{n}{n-1}$$

5. (Putnam preparation) Find the limit

$$\lim_{x \rightarrow 0} \frac{1 - (\cos x)^{\sin x}}{x^3}$$

6. (IrMO 2019) Marius is going to sports camp for 7 days. Each day, he will play exactly one of three sports; hockey, tennis or camogie. The only restriction is that in any period of 4 consecutive days, he must play all three sports.
Find the number of possible sports schedules for Marius' week.
7. (Canadian IMO 1988) A house has the shape of a triangle of perimeter P and area A . The garden consists of all the land within distance 5 of the house. How much land do the garden and house together occupy?
8. (Classic Problem) Show that there exists no integers $n > 1$, for which $n \mid 2^n - 1$.
9. (Berkeley problems in mathematics) Let $f(x), 0 \leq x \leq 1$, be a real-valued continuous function. Show that

$$\lim_{n \rightarrow \infty} (n+1) \int_0^1 x^n f(x) dx = f(1)$$

10. (Putnam 1987) Let us define the following regions in the plane \mathbb{R}^2 :

$$A = \left\{ (x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\}$$

$$B = \left\{ (x, y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\}$$

$$C = \{ (x, y) : x^3 - 3xy^2 + 3y = 1 \}$$

$$D = \{ (x, y) : 3x^2y - 3x - y^3 = 0 \}$$

Prove that $A \cap B = C \cap D$.

Weekly Problem: In how many ways is it possible to fill the squares of a chessboard with either 1 or -1 such that the sum of the elements in each 2×2 subarray is 0?