Mathsoc Problem Solving / Puzzle League 2019/2020

Week 4

07/10/2019

1. Prove that $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$. Bonus points if you can prove this with combinatorics. Hint available

2. (*Berkeley Problems in Mathematics*) Let A be an $n \times n$ matrix, with $n >!$ such that $A^n = 0$, but $A^{n-1} \neq 0$. Prove that there does not exist a matrix E such that $E^2 = A$.

3. (BMO 2016 paper 1) Sinéad and Sergey play a game, with Sergey going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Which of the players, if either, has a winning strategy? hint available
The numbers which are not expressible as the difference of two nonzerosquares are all numbers of the form $4n + 2$.

4. (Classic problem) The physics department at Trinity College Letterkenny employs 100 unscrupulous academics. Each academic has plagiarised their most recent work; for every given academic the rest of the staff are aware that the academic has plagiarised their work, but the academic does not know that everyone else is aware. Then one day, a visiting lecturer from Undivided IT arrives and at an all-staff meeting (ie everyone is present) he makes the shock announcement: "I regret to announce that at least one of you has plagiarised. We shall meet everyday for the forseeable future, beginning tomorrow, until all faculty members guilty of plagiarism have resigned. "
Then, exactly 100 days later, every single physics academic resigns. Why? Hint available

5. (*Berkeley Problems in Mathematics*) Let f be a real-valued continuous function on $[0, +\infty)$ such that

$$\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\,\mathrm{d}t \right)$$

exists. Prove that $\lim_{x\to\infty} f(x) = 0$.

6. (Putnam preparation) Does there exist an $n \times n$ real matrix such that $\mathrm{tr}(A) = 0$ and $A^2 + A^T = I$? Here $\mathrm{tr}(A)$ denotes the sum of the diagonal entries of A.

7. (Irish Mathematical Olympiad 2011) Suppose $abc \neq 0$. Express in terms of a, b, c the solutions x, y, z, u, v, w of the following equations:
$x + y = a$ $z + u = b$ $v + w = c$ $ay = bz$ $ub = cv$ $wc = ax$

8. (Stolen from Vladimir Dotsenko) Let A be an $n \times n$ matrix with entries $a_{ij} = \gcd(i, j)$. Prove that $\text{Det } A = \phi(1)\phi(2)\cdots\phi(n)$, where $\phi(n)$ is the Euler totient function, the number of positive integers less than or equal to n that are coprime to n (and $\phi(1) = 1$).

9. (Putnam preparation) If a, b, c are positive real numbers prove that

$$(a + b - c)(c + a - b)(b + c - a) \leq abc$$

10. (International Mathematics Competition 2019) A four-digit number YEAR is called *very good* if the system of equations

$$\begin{align} Yx + Ey + Az + Rw &= Y \tag{1}\\ Rx + Yy + Ez + Aw &= E \tag{2}\\ Ax + Ry + Yz + Ew &= A \tag{3}\\ Ex + Ay + Rz + Yw &= R \tag{4}\\ & \tag{5} \end{align}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century, including 2100 but excluding 2000.

Weekly Problem: Find all primes p such that $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

1 Hints / spoilers section

- (Problem 1) Think about problem 1 from last week; is there more than one way of counting subsets?
- (Problem 3) The numbers which are not expressible as the difference of two squares are all numbers of the form $4n + 2$.
- (Problem 4) Think about small cases first, then you can use induction to explain the general case. All the physicists are perfect logical thinkers, oddly.