Mathsoc Problem Solving 2019/2020

Week 2

23/09/2019

1. (UCC Maths enrichment) Prove that if a set has n different elements, then the number of subsets (including the empty set!) is 2^n.

2. (Putnam preparation) Let $A = \{1, 2, 3, 4, 5\}$ and define the operation $\circ$ on A as $a \circ b = a$. Prove that $\circ$ is associative.

3. (UCD Maths enrichment) Consider an 8×8 chessboard and remove two diametrically opposite corner unit squares. Is it possible to cover (without overlapping) the remaining 62 unit squares with dominoes?

4. (*Irish Mathematical Olympiad 1996*) Let $f(x)$ be a function from $[0, 1]$ to the set of all real numbers such that:

   (a) $f(1) = 1$

   (b) $f(x) \geq 0$ for all x

   (c) $f(x + y) \leq f(x) + f(y)$ whenever x, y, and $x + y$ are all in $[0, 1]$.

   Prove that $f(x) \leq 2x$.

5. (*Berkeley Problems in Mathematics*) Suppose f is a continuous real-valued function. Show that there exists φ with $0 \leq \varphi \leq 1$ such that

$$\int_0^1 f(x)x^2 \, \mathrm{d}x = \frac{1}{3} f(\varphi)$$

6. (1999 Romanian mathematical Olympiad) Let a, b, c be nonzero integers with $a \neq c$ and

$$\frac{a}{c} = \frac{a^2 + b^2}{c^2 + b^2}$$

   Prove that $a^2 + b^2 + c^2$ can never be a prime.

7. (EGMO selection test 2019) Let a, b, c be sides of a triangle. Prove that

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{b + a} < 2$$

8. (*Norwegain Mathematical Olympiad 1995*) Prove that if $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1$ for real numbers x, y then $x+y=0$.

9. (*IMO 2019*) The Bank of Hamiltonia issues coins with a H on one side and a T on the other. Fergal has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k>0$ coins showing H, then he turns over the kth coin from the left; otherwise, all coins show T and he stops. For example, if $n=3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

   (a) Show that, for each initial configuration, Harry stops after a finite number of operations.

   (b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT)=3$ and $L(TTT)=0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

10. (Putnam preparation) Find the following limit:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$$

Weekly Problem: Find all polynomials $f(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$ such that $a_j=\pm1$ for $j=1,2,\ldots,n$ and every root of f is real.