Mathsoc Problem Solving 2019/2020

Week 2

23/09/2019

1. - Prove that for all natural numbers n, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$
   - Now prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

2. (*Trinidad and Tobago selection test 2016*) Let P, Q, R be points on the sides AB, BC and CA respectively, of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at the points C and A intersect the lines RQ and RP at the points X and Y, respectively.
   Prove that $RX = RY$.

3. (*Problem solving strategies, Arthur Engel*) $2n$ players are participating in a tennis tournament. Find the number P_n of pairings for the first round.

4. (*Irish Mathematical Olympiad 2017*) A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \ldots, B_{n-1}$ and A is a point such that $\angle B_0AB_n$ is a right angle. Prove that

$$\sum_{k=0}^{n} |AB_k|^2 = \sum_{k=0}^{n} |B_0B_k|^2$$

5. (*Berkeley Problems in Mathematics*) Let $f(x)$ be a real-valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f(x)^2}$$

   Show that $\lim_{x\to\infty} f(x)$ exists, and is less than $1 + \dfrac{\pi}{4}$

6. (Josephus Problem, also featured on Numberphile) n people stand in a circle, waiting to be executed. The first person is killed. After that, it goes clockwise, the first alive person is spared but the second alive person is killed; continuing until there is only one person left.
   Where should a person stand in the circle to ensure that they are the last surviving person?

7. (Putnam preparation) Let A, B be two $n \times n$ matrices such that $AB = BA$. Prove that if $\mathbf{v}$ is an eigenvector for A, then $B\mathbf{v}$ is an eigenvector for A as well.
   (v is an eigenvector of a matrix A if there exists a constant λ such that $Av = \lambda v$)

8. (*French IMO selection test 2003*) Find the minimum value of $a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3$ over all permutations $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ of $\{1, 2, \ldots, 9\}$

9. (*IMO 2019*) Find all pairs of positive integers (k, n) satisfyig the following equation:

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$$

10. (Putnam 2007) Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \dfrac{1}{24}$ and $x = \alpha y^2 + \alpha y + \dfrac{1}{24}$ are tangent to each other.

Weekly Problem: Let $f(x)$ be a continuous real-valued function defined on $[0, 1]$. Find the maximum value of the following expression:

$$\int_0^1 x^2 f(x) - x f(x)^2 \mathrm{d}x$$

Try to solve it without using Euler-Lagrange!