Mathsoc Problem Solving 2019/2020

Week 1

16/09/2019

1. Prove that for all positive real numbers x, y that $\frac{x+y}{2} \geq \sqrt{xy}$

2. (EGMO selection test 2012) We are given a set X containing 100 integers, none of which is divisible by 3. We are asked to carry out the following task: choose 7 integers from this set so that for any pair of integers x and y we choose, the difference $x - y$ is not divisible by 9.
   - Prove that this task is impossible.
   - If we are instead asked to choose 6 integers from X, is the task always possible?

3. Raynor the snail wishes to travel from one corner of an 8 by 8 chessboard to the diagonally opposite corner. He can only travel one square at a time to a square that is vertically or horizontally adjacent to the current one he is on. ?In how many ways can this be done?
   - Generalise this problem to an m by n board!

4. (Putnam preparation) If $ABCD$ is a parralelogram, prove that $AB^2+BC^2+CD^2+DA^2 = AC^2+BD^2$ (AB denotes the distance between A and B. Also A and C are diagonally opposite, as are B and D.

5. (UCD maths enrichment) Define a sequence by $S_1 = S_2 = 1$, and $S_n = S_{n-1}^2 + S_{n-2}^2$. Show that S_n is never divisible by 7.

6. (continued) Show that $S_n \leq 2^{(2^{n-2.7})}$

7. (Putnam preparation) Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function and let $\{a_n\}_{n\geq 1}$ be a sequence in $[a, b]$. Prove that there exists $x \in [a, b]$ such that $f(x) = \sum_{n=0}^{\infty} \frac{f(a_n)}{2^n}$

8. (BMO 2016) Consecutive positive integers $m, m + 1, m + 2$ and $m + 3$ are divisible by consecutive odd positive integers $n, n + 2, n + 4$ and $n + 6$ respectively. Determine the smallest possible m in terms of n.

9. (IMO 2019) Find all functions mapping from the integers to the integers satisfying, for all $m, n \in \mathbb{Z}$,

$$f(2m) + 2f(n) = f(f(m + n))$$

10. (Putnam 2004) Basketball star Jan Manschot's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when S(N) was exactly $80\% of N$?

Weekly Problem: Find a ten-digit positive integer in base 10 which 'describes itself'. That is, a number as follows: the first digit tells us how many 0's are in the number, the second digit tells us how many 1's are in the number, and so on until the final digit tells us how many 9's are in the number.