Problem Solving 2018/19

Week 15

1. Find the least positive integer n such that the prime factorizations of $n, n+1$, and $n+2$ each have exactly two factors (as 4 and 6 do, but 12 does not).

(Purple Comet 2014 P4)

2. The edges of a cube are coloured with three colours such that each vertex is the endpoint of an edge of each of the three colours. Show that there are four parallel edges of the same colour.

(EGMO Selection Test 2019 Q1)

3. Let a, b, c be the sides of a triangle. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

(EGMO Selection Test 2019 Q3)

4. Let S be a set of $6n$ points on a line. $4n$ of these points are painted blue and the other $2n$ points are painted green. Prove that there exists a line segment that contains exactly $3n$ points from S, such that $2n$ of them are blue and the other n are green.

(EGMO Selection Test 2019 Q8)

5. Find all triples $(a,\ b,\ c)$ of positive integers such that

$$a^2 + b + 3 = (b^2 - c^2)^2.$$

(Japan 2019 P1)

6. In a town with n people, m clubs have been formed. Every club has an odd number of members, and every two clubs have an even number of members in common. Prove that $m \leq n$.

(Oddtown Theorem) (Yufei Zhao LA P8)

7. Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute ($n = 1, 2, \dots$) each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

(Balkan Maths Olympiad 2018 P2)

8. Let $f_1, f_2, \cdots$ be continuous functions on $[0, 1]$ satisfying $f_1 > f_2 > \cdots$ and such that $\lim_{n\to\infty} f_n(x) = 0$ for each x. Must the sequence (f_n) converge to 0 uniformly on $[0, 1]$?

(Berkeley Problems in Mathematics 1.6.7)

9. Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for all $g \in \mathcal{A}$ with $g \neq f$. Prove that $f(T) = T$.

(IMO 2017 SL A3)

Problem of the Week

A 3 by 3 matrix has three entries equal to 2, three entries equal to 5, and three entries equal to 8. Find the maximum possible value of the determinant.

(Purple Comet 2011 P13)