Problem Solving 2018/19

Week 14

1. Find the remainder when dividing $a = 1^{2019} + 2^{2019} + 3^{2019} + \cdots + 2019^{2019}$ by 5.

(UCC Maths Enrichment)

2. Let $S = x_1, \ldots, x_n$ be a set of points in the plane such that the distance between any two points is at least 1. Show that there are at most $3n$ pairs of points at distance exactly 1.

(UCC Maths Enrichment)

3. If $f, g : \mathbb{R} \to \mathbb{R}$ are functions that satisfy $f(x+g(y)) = 2x+y \ \forall x, y \in \mathbb{R}$, then determine $g(x + f(y))$.

(Flanders 2007 Q4)

4. Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$.

(Berkeley Problems in Mathematics 7.5.10)

5. For some integer n, a set of n^2 magical chess pieces arrange themselves on a square $n^2 \times n^2$ chessboard composed of n^4 unit squares. At a signal, the chess pieces all teleport to another square of the chessboard such that the distance between the centres of their old and new squares is n. The chess pieces win if, both before and after the signal, there are no two chess pieces in the same row or column. For which values of n can the chess pieces win?

(BMO Round 2 P2)

6. Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets (equivalently, open intervals to open intervals) must be monotonic (increasing or decreasing).

(Berkeley Problems in Mathematics 1.2.8)

7. Prove that if G is a group containing no subgroup of index 2, then any subgroup of index 3 in G is a normal subgroup.

(Berkeley Problems in Mathematics 6.4.12)

8. Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^m x = 0, T^{m-1}x \neq 0$ for some positive integer m. Show that $x, Tx, \ldots, T^{m-1}x$ are linearly independent.

(Berkeley Problems in Mathematics 7.1.10)

9. Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

(Putnam 2018 B6)

Problem of the Week

Let N be a positive integer whose digits add up to 23. What is the greatest possible product the digits of N can have?

(Purple Comet 2012 Problem 15)