Problem Solving 2018/19

Week 13

1. How many squares (of any size) are there on a chessboard?

2. Minimize $\frac{6x}{y} + \frac{12y}{z} + \frac{3z}{x}$ for $x, y, z > 0$.

(UCD Maths Enrichment)

3. A positive integer is said to be *near-square* if it is a product of two positive integers differing by 1. For example, 20 is a near-square because $20 = 4 \times 5$. Prove that every near-square positive integer can be expressed as the ratio of two other near-square positive integers.

(EGMO Selection Test 2017)

4. Consider an alphabetized list of all the arrangements of the letters in the word BETWEEN. Then BEEENTW would be in position 1 in the list, BEEENWT would be in position 2 in the list, and so forth. Find the position that BETWEEN would be in the list.

(Purple Comet 2017 Problem 7)

5. Give an example of a continuous function $v : \mathbb{R} \to \mathbb{R}^3$ with the property that $v(t_1), v(t_2)$, and $v(t_3)$ form a basis for $\mathbb{R}^3$ whenever t_1, t_2, and t_3 are distinct points of $\mathbb{R}$.

(Berkeley Problems in Mathematics 7.2.9)

6. Let p be an odd prime. How many non-empty subsets of

$$\{1, 2, \ldots, p-2, p-1\}$$

have a sum which is divisible by p?

(BMO Round 2 P3)

7. Find all functions f from the posituve real numbers to the positive real numbers for which $f(x) \leq f(y)$ whenever $x \leq y$ and

$$f(x^4) + f(x^2) + f(x) + f(1) = x^4 + x^2 + x + 1$$

for all $x > 0$.

(BMO Round 2 P4)

8. Let m and n be positive integers with $\gcd(m, n) = 1$, and let

$$a_k = \left\lfloor \frac{mk}{n} \right\rfloor - \left\lfloor \frac{m(k-1)}{n} \right\rfloor$$

for $k = 1, 2, \ldots, n$. Suppose that g and h are elements in a group G and that

$$gh^{a_1}gh^{a_2} \cdots gh^{a_n} = e,$$

where e is the identity element. Show that $gh = hg$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

(Putnam 2018 A4)

9. Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

(Putnam 2018 A6)

Problem of the Week

Show that the sum of squares of 5 consecutive numbers cannot be a perfect square.

(UCC Maths Enrichment)