Problem Solving 2018/19

Week 12

1. Find the least positive integer m such that $\mathrm{lcm}(15, m) = \mathrm{lcm}(42, m)$. Here $\mathrm{lcm}(a, b)$ is the least common multiple of a and b.

(Purple Comet 2017 Problem 4)

2. The numbers $1, 2, \ldots$ are placed in a triangle as following:

$$\begin{array}{llll} 1 & & & \\ 2 & 3 & & \\ 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 \\ \vdots & & & \end{array}$$

What is the sum of the numbers on the n-th row?

(Flanders 2007 Q1)

3. Prove that $5n^3 + 3n^2 + 4n$ is a multiple of 6 for all natural numbers n.

(UCC Maths Enrichment)

4. Alice and Bob play a game with a string of 2017 pearls. In the first move, Alice cuts the string and Bob chooses a part. Thereafter, the player who chose a part at the end of a move will cut the string in the next move. A player loses if he or she obtains a string with a single pearl such that no more cuts are possible. Which of the two players has a winning strategy?

(Ireland EGMO Selection Test 2018 Q3)

5. We have a deck of 10,000 cards, numbered from 1 to 10,000. A step consists of removing every card which has a perfect square on it, and then renumbering the remaining cards, starting from 1, in a consecutive way (i.e., numbering them 1, 2, 3, etc.) Find, with proof, the number of steps needed to remove all but one card.

(UCD Maths Enrichment)

6. Let G be a group. For any subset X of G, define its centralizer $C(X)$ to be $\{y \in G : xy = yx, \forall x \in X\}$. Prove the following:

   (a) If $X \subset Y$, then $C(Y) \subset C(X)$.

   (b) $X \subset C(C(X))$.

   (c) $C(X) = C(C(C(X)))$.

(Berkeley Problems in Mathematics 6.1.11)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

(Putnam 2018 A5)

8. Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, 0 \leq a \leq 2, 0 \leq b \leq 100, \text{ and } a, b \in \mathbb{Z} \right\}.$$

   Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

(Putnam 2018 B1)

9. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

(Putnam 2018 B4)

Problem of the Week

Find the number of positive integers less than or equal to 2019 that have at least one pair of adjacent digits that are both even. For example, count the numbers 24, 1862, and 2012, but not 4, 58, or 1276.

(Purple Comet 2017 Problem 10)