Problem Solving 2018/19

Week 11

1. A jar contains one white marble, two blue marbles, three red marbles, and four green marbles. If you select two of these marbles without replacement, what is the probability that both marbles will be the same colour?

(Purple Comet 2010 Problem 11)

2. Show that among 19 consecutive natural numbers there exists one smaller than the sum of its divisors (not including 1 and itself).

(UCC Maths Enrichment)

3. Let Γ be a semicircle with diameter AB. The point C lies on the diameter AB and points E and D lie on the arc BA, with E between B and D. Let the tangents to Γ at D and E meet at F. Suppose that $\angle ACD = \angle ECB$.
Prove that $\angle EFD = \angle ACD + \angle ECB$.

(British Maths Olympiad 2018 Paper 1 Q4)

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(1) = 1$ and

$$f(n) = n - f(f(n-1)), \quad \forall n \geq 2.$$

Prove that $f(n + f(n)) = n$ for each positive integer n.

(Bulgaria 2010 P2 Q2)

5. An accurate map of Ireland is spread out flat on a table in the MathSoc Room. Prove that there is exactly one point on the map lying directly over the point it represents.

(Berkeley Problems in Mathematics 4.3.1)

6. Esmeralda writes $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, all belonging to the interval $[0, 1]$, around a circle and multiplies all the pairs of numbers neighboring to each other, obtaining, in the counterclockwise direction, the products $p_1 = x_1x_2$, $p_2 = x_2x_3$, $\ldots$, $p_{2n} = x_{2n}x_1$. She adds the products with even indices and subtracts the products with odd indices. What is the maximum possible number Esmeralda can get?

(Brazil 2018 Q4)

7. Let $X \subset \mathbb{R}^n$ be a closed set and r a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n : |x - y| = r \text{ for some } x \in X\}$. Show that Y is closed.

(Berkeley Problems in Mathematics 4.1.8)

8. Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

(Putnam 2018 A2)

9. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.

(Putnam 2018 B3)

Problem of the Week

Two solid cylinders are mathematically similar. The sum of their heights is 1. The sum of their surface areas is 8π. The sum of their volumes is 2π. Find all possibilities for the dimensions of each cylinder.

(British Maths Olympiad 2018 Paper 1 Q5)