Problem Solving 2018/19

Week 9

1. Show that there are 1000 consecutive numbers all of which are not prime.

   (UCC Maths Enrichment)

2. 21 people are taking part in a chess tournament are to be split into two teams. Each player will play every player on the opposing team exactly once, and will not play any players on his or her own team. What is the greatest number of matches that can be played, in this way?

   (UCC Maths Enrichment)

3. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that for all real numbers x and y satisfy

$$f(x+y)f(xy) = f(x^2 - y^2 + 1).$$

   (Estonia 16/17 O11)

4. Ares multiplies two integers which differ by 9. Grace multiplies two integers which differ by 6. They obtain the same product T. Determine all possible values of T.

   (British Maths Olympiad 2018 Paper 1 Q3)

5. Prove that $(\cos\theta)^p \leq \cos(p\theta)$ for $0 \leq \theta \leq \pi/2$ and $0 < p < 1$.

   (Berkeley Problems in Mathematics 1.1.1)

6. We say that a rectangle with side lengths a and b *fits inside* a rectangle with side lengths c and d if either ($a \leq c$ and $b \leq d$) or ($a \leq d$ and $b \leq c$). For instance, a rectangle with side lengths 1 and 5 fits inside another rectangle with side lengths 1 and 5, and also fits inside a rectangle with side lengths 6 and 2.
   Suppose S is a set of 2019 rectangles, all with integer side lengths between 1 and 2018 inclusive. Show that there are three rectangles A, B, and C in S such that A fits inside B, and B fits inside C.

   (IrMO 2018, P1 Q4)

7. Consider a graph with n vertices, each of which can be in one of two states, on or off. A move consists of selecting a vertex and changing its state and the state of every vertex adjacent to it. Prove that if all the vertices are initially off there is a sequence of moves that turns all the vertices on.

8. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

(Putnam 2018 A1)

9. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

(Putnam 2018 B2)

Problem of the Week

A list of five two-digit positive integers is written in increasing order on a blackboard. Each of the five integers is a multiple of 3, and each digit 0,1,2,3,4,5,6,7,8,9 appears exactly once on the blackboard. In how many ways can this be done? Note that a two-digit number cannot begin with the digit 0.

(British Maths Olympiad 2018 Paper 1 Q1)