Problem Solving 2018/19

Week 10

1. (a) Which is the larger number: $A = 200!$ or $B = 100^{200}$?
   (b) Which is the larger number: $A = 2000!$ or $B = 100^{2000}$? Justify your answer.

(UCD Maths Enrichment)

2. How many rectangles (of any size) are there on a chessboard?

3. Let P be a polynomial satisfying $P(x+1)+P(x-1) = x^3$ for all real numbers x. Find the value of $P(12)$.

(Purple Comet 2017 Problem 12)

4. Let $[n] = \{1, 2, ..., n\}$ and consider a k-tuple $A_1, A_2, \ldots, A_k$ of disjoint subsets of $[n]$ whose union is the entire set $[n]$.
   (a) Find the number k-tuples as above, if empty subsets are allowed.
   (b) Find the number k-tuples as above, if empty subsets are not allowed.
   (c) Let $S(n, k)$ denote the number of ways to partition a set of n elements into k disjoint, non-empty subsets (the order of the subsets is not important). Write a formula for $S(n, k)$.
   (d) Find the number of surjective functions from $[n]$ to a set with k elements.

(UCC Maths Enrichment)

5. Ada the ant starts at a point O on a plane. At the start of each minute she chooses North, South, East or West, and marches 1 metre in that direction. At the end of 2018 minutes she finds herself back at O. Let n be the number of possible journeys which she could have made. What is the highest power of 10 which divides n?

(British Maths Olympiad 2018 Paper 1 Q6)

6. A sequence $a_1, a_2, \ldots, a_n, \ldots$ of natural numbers is defined by the rule

$$a_{n+1} = a_n + b_n \ (n = 1, 2, \ldots)$$

where b_n is the last digit of a_n. Prove that such a sequence contains infinitely many powers of 2 if and only if a_1 is not divisible by 5.

(Benelux Olympiad 2012 P1)

7. Suppose that f is a continuous function on $\mathbb{R}$ which is periodic with period 1, i.e., $f(x+1) = f(x)$. Show:

   (a) The function f is bounded above and below and achieves its maximum and minimum.

   (b) The function f is uniformly continuous on $\mathbb{R}$.

   (c) There exists a real number x_0 such that

$$f(x_0 + \pi) = f(x_0).$$

(Berkeley Problems in Mathematics 1.2.3)

8. Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

(Putnam 2018 A3)

9. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

(Putnam 2018 B5)

Problem of the Week

For each positive integer $n \geq 3$, we define an n-ring to be a circular arrangement of n (not necessarily different) positive integers such that the product of every three neighbouring integers is n. Determine the number of integers n in the range $3 \leq n \leq 2018$ for which it is possible to form an n-ring.

(British Maths Olympiad 2018 Paper 1 Q2)